M. Sc. (Mathematics) **Semester-I**

COMPLEX ANALYSIS

Paper Code: 20MAT21C4

DIRECTORATE OF DISTANCE EDUCATION
MAHARSHI DAYANAND UNIVERSITY, ROHTAK
(A State University established under Haryana Act No. XXV of 1975)
NAAC 'A+' Grade Accredited University

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MASTER OF SCIENCE (MATHAMATICS)
First Semester
Paper code: 20MAT21C4
Complex Analysis

M. Marks = 100
Term End Examination = 80
Assignment = 20
Time = 3 hrs

Course Outcomes

Students would be able to:

CO1 **Be familiar with complex numbers and their geometrical interpretations.**

CO2 **Understand the concept of complex numbers as an extension of the real numbers.**

CO3 Represent the sum function of a power series as an analytic function.

CO4 Demonstrate the ideas of complex differentiation and integration for solving related problems and establishing theoretical results.

CO5 Understand concept of residues, evaluate contour integrals and solve polynomial equations.

Section - I

Function of a complex variable, Continuity, Differentiability, Analytic functions and their properties, Cauchy-Riemann equations in cartesian and polar coordinates, Power series, Radius of convergence, Differentiability of sum function of a power series, Branches of many valued functions with special reference to argz, logz and za.

Section - II

Path in a region, Contour, Complex integration, Cauchy theorem, Cauchy integral formula, Extension of Cauchy integral formula for multiple connected domain, Poisson integral formula, Higher order derivatives, Complex integral as a function of its upper limit, Morera theorem, Cauchy inequality, Liouville theorem, Taylor theorem.

Section - III

Zeros of an analytic function, Laurent series, Isolated singularities, Cassorati-Weierstrass theorem, Limit point of zeros and poles. Maximum modulus principle, Schwarz lemma, Meromorphic functions, Argument principle, Rouche theorem, Fundamental theorem of algebra, Inverse function theorem.

Section - IV

Calculus of residues, Cauchy residue theorem, Evaluation of integrals of the types $\int_0^{2\pi} f(cos\theta, sin\,\theta)d\theta$ $\int_{-\infty}^{\infty} f(x)dx$, $\int_0^{\infty} f\left(x\right)\sin mx\,dx$ and $\int_0^{\infty} f\left(x\right)\cos mx\,dx$, Conformal mappings. Space of analytic functions and their completeness, Hurwitz theorem, Montel theorem, Riemann mapping theorem.

Note : The question paper of each course will consist of **five** Sections. Each of the sections **I to IV** will contain **two** questions and the students shall be asked to attempt **one** question from each. **Section-V** shall be **compulsory** and will contain **eight** short answer type questions without any internal choice covering the entire syllabus.

Books Recommended:

1. H.A. Priestly, Introduction to Complex Analysis, Clarendon Press, Oxford, 1990.
2. J.B. Conway, Functions of One Complex Variable, Springer-Verlag, International student-Edition, Narosa Publishing House, 2002.
3. Liang-Shin Hann & Bernand Epstein, Classical Complex Analysis, Jones and Bartlett Publishers International, London, 1996.
4. E.T. Copson, An Introduction to the Theory of Functions of a Complex Variable, Oxford University Press, London, 1972.
5. E.C. Titchmarsh, The Theory of Functions, Oxford University Press, London.
6. Ruel V. Churchill and James Ward Brown, Complex Variables and Applications, McGraw-Hill Publishing Company, 2009.
7. H.S. Kasana, Complex Variable Theory and Applications, PHI Learning Private Ltd, 2011.
8. Dennis G. Zill and Patrik D. Shanahan, A First Course in Complex Analysis withApplications, John Bartlett Publication, 2nd Edition, 2010.

Contents

SECTION-I

1.1 Complex Number: A complex number is a number that can be expressed in the form $x + iy$, where x and y are real numbers. If we write $z = x + iy$, where x and y are real variable's then z is called a complex variable.

It is clear that the set of complex numbers includes the real numbers as a subset. When real number x is displayed as point $(x, 0)$ on the real axis and Complex number of the form $(0, y)$ correspond to point on the y axis and is called purely imaginary number, when $y \neq 0$. The y axis is then referred to as the imaginary axis. It is customary to denote a complex number (x, y) by z (see Fig. 1).

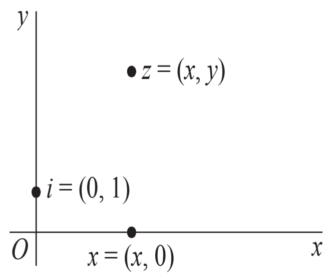


Figure 1

To each complex number there corresponds one and only one point in the xy-plane and conversely, to each point in the xy-plane there exists one and only one complex number. xy-plane is also called complex plane, Argand plane and Gaussian plane.

The real numbers x and y are, moreover, known as the real and imaginary parts of z, respectively; and we write

$$x = Re\, z,\ y = Im\, z. \tag{1}$$

The sum $z_1 + z_2$ and product z_1z_2 of two complex numbers

$$z_1 = (x_1, y_1) \text{ and } z_2 = (x_2, y_2) \tag{2}$$

are defined as follows:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \tag{3}$$

$$(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, y_1x_2 + x_1y_2). \tag{4}$$

Note that the operations defined by equations (3) and (4) become the usual operations of addition and multiplication when restricted to the real numbers and

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0),$$

$$(x_1, 0)(x_2, 0) = (x_1x_2, 0).$$

The complex number system is, therefore, a natural extension of the real number system.

1.1.1 Basic algebraic properties of complex variable: Various properties of addition and multiplication of complex numbers are the same as for real numbers. We list here the more basic of these algebraic properties and verify some of them.

The **commutative laws**

$$z_1 + z_2 = z_2 + z_1,\ z_1 z_2 = z_2 z_1$$

and the **associative laws**

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3),\ (z_1 z_2) z_3 = z_1 (z_2 z_3),$$

follow easily from the definitions of addition and multiplication of complex numbers and the fact that real numbers obey these laws.

For example, if

$z_1 = (x_1, y_1)$ *and* $z_2 = (x_2, y_2)$, then

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2) = (x_2 + x_1, y_2 + y_1) = z_2 + z_1.$$

Verification of the rest of the above laws, as well as the **distributive law**

$z(z_1 + z_2) = zz_1 + zz_2$, is similar.

According to the commutative law for multiplication, $iy = yi$, one can write $z = x + yi$ instead of $z = x + iy$. Also, because of the associative laws, a sum $z_1 + z_2 + z_3$ or a product $z_1 z_2 z_3$ is well defined without parentheses, as is the case with real numbers.

The **additive identity** $0 = (0, 0)$ and the multiplicative identity $1 = (1, 0)$ for real numbers carry over to the entire complex number system. That is,

$$z + 0 = z \text{ and } z \cdot 1 = z \tag{5}$$

for every complex number z. Furthermore, 0 and 1 are the only complex numbers with such properties. There is associated with each complex number $z = (x, y)$ an **additive inverse**

$$-z = (-x, -y), \tag{6}$$

satisfying the equation $z + (-z) = 0$. Moreover, there is only one additive inverse for any given z. Since the equation

$$(x, y) + (u, v) = (0 , 0)$$

implies that

$$u = -x \text{ and } v = -y.$$

For any nonzero complex number $z = (x, y)$, there is a number z^{-1} such that $zz^{-1} = 1$. This **multiplicative inverse** is less obvious than the additive one. To find it, we seek real numbers u and v, expressed in terms of x and y, such that

$$(x, y)\,(u, v) = (1, 0).$$

According to equation (4), which defines the product of two complex numbers, u and v must satisfy the pair

$$xu - yv = 1,\ yu + xv = 0$$

of linear simultaneous equations and simple computation yields the unique solution

$$u = \frac{x}{x^2 + y^2},\ v = \frac{-y}{x^2 + y^2}$$

So, the multiplicative inverse of $z = (x, y)$ is

$$z^{-1} = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right) (z \neq 0). \tag{7}$$

The inverse z^{-1} is not defined when $z = 0$. In fact, $z = 0$ means that $x^2 + y^2 = 0$ and this is not permitted in expression (7). Such properties continue to be anticipated because they also apply to real numbers. We begin with the observation that the existence of multiplicative inverse enables us to show that if a product $z_1 z_2$ is zero, then so is, at least one of the factors z_1 and z_2. For suppose that $z_1 z_2 = 0$ and $z_1 \neq 0$. The inverse z_1^{-1} exists and any complex number times zero is zero. Hence,

$$z_2 = z_2.1 = z_2\left(z_1 z_1^{-1}\right) = \left(z_1^{-1} z_1\right) z_2 = z_1^{-1}\left(z_1 z_2\right) = z_1^{-1}.0 = 0.$$

That is, if $z_1 z_2 = 0$, either $z_1 = 0$ or $z_2 = 0$; or possibly both of the numbers z_1 and z_2 are zero. Another way to state this result is that if two complex numbers z_1 and z_2 are nonzero, then so is their product $z_1 z_2$.

In terms of additive and multiplicative inverses, Subtraction and division are defined as

$$z_1 - z_2 = z_1 + (-z_2), \tag{8}$$

$$\frac{z_1}{z_2} = z_1 z_2^{-1} \qquad (z_2 \neq 0). \tag{9}$$

Thus, in view of expressions (8) and (9)

$$z_1 - z_2 = (x_1, y_1) + (-x_2, -y_2) = (x_1 - x_2, y_1 - y_2) \tag{10}$$

and

$$\frac{z_1}{z_2} = (x_1, y_1)\left(\frac{x_2}{x_2^2 + y_2^2}, \frac{-y_2}{x_2^2 + y_2^2} \right) = \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}, \frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2} \right), (z_2 \neq 0) \tag{11}$$

where $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$.

Using $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, one can write expressions (10) and (11) here as

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2) \tag{12}$$

and

$$\frac{z_1}{z_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\,\frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2} \qquad (z_2 \neq 0) \tag{13}$$

or

$$\frac{z_1}{z_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)}. \tag{14}$$

1.1.2 Example: Show that

(i) $|z_1 + z_2| \leq |z_1| + |z_2|$

(ii) $||z_1| - |z_2|| \leq |z_1 - z_2|$

Solution: To prove these, we first observe that if $z_1 + z_2 = 0$, then there is nothing to prove. If $z_1 + z_2 \neq 0$, then $|z_1 + z_2| \neq 0$. Since $\operatorname{Re} z \leq |z|$, we have

$$\left|\frac{z_1}{z_1+z_2}\right|+\left|\frac{z_2}{z_1+z_2}\right| \geq \operatorname{Re}\left(\frac{z_1}{z_1+z_2}\right)+\operatorname{Re}\left(\frac{z_2}{z_1+z_2}\right)=1$$

From which (i) follows.

To prove the second inequality, we write $z_1 = z_2 + (z_1 - z_2)$ so that, by (i),

$$|z_1| \leq |z_2| + |z_1 - z_2|,$$

i.e. $$|z_1| - |z_2| \leq |z_1 - z_2|.$$

Similarly, we obtain

$$|z_2| - |z_1| \leq |z_1 - z_2| = |z_1 - z_2|.$$

On combining these two inequalities, we get (ii).

1.1.3 Vector and moduli: It is natural to associate any nonzero complex number $z = x + iy$ with the directed line segment, or vector, from the origin to the point (x, y) that represents z in the complex plane. In fact, we often refer to z as the point z or the vector z. In Fig. 2 the numbers $z = x + iy$ and $-2 + i$ are displayed graphically as both points and radius vectors.

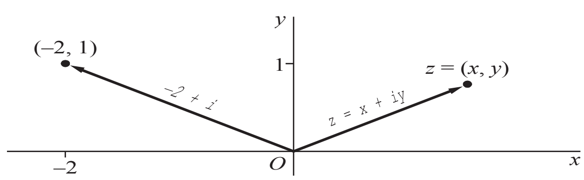


Figure 2

When $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the sum

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

corresponds to the point $(x_1 + x_2, y_1 + y_2)$. It also corresponds to a vector with those coordinates as its components. Hence $z_1 + z_2$ may be obtained vectorially as shown in Fig.3.

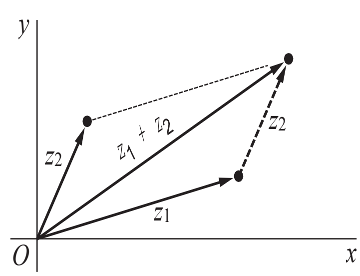


Figure 3

Although the product of two complex numbers z_1 and z_2 is itself a complex number represented by a vector, that vector lies in the same plane as the vectors for z_1 and z_2. Evidently, this product is neither the scalar nor the vector product used in ordinary vector analysis.

The vector interpretation of complex numbers is especially helpful in extending the concept of absolute values of real numbers to the complex plane. The modulus, or absolute value, of a complex number z

= $x + iy$ is defined as the nonnegative real number $\sqrt{x^2+y^2}$ and is denoted by $|z|$, that is, $|z| = \sqrt{x^2+y^2}$.

Geometrically, the number $|z|$ is the distance between the point (x, y) and the origin, or the length of the radius vector representing z. It reduces to the usual absolute value in the real number system when $y = 0$. Note that while the inequality $z_1 < z_2$ is meaningless unless both z1 and z2 are real, the statement $|z_1| < |z_2|$ means that the point z_1 is closer to the origin than the point z_2. Since $|-3 + 2i| = \sqrt{13}$ and $|1 + 4i| = \sqrt{17}$, we know that the point $-3 + 2i$ is closer to the origin than $1+ 4i$.

The distance between two points (x_1, y_1) and (x_2, y_2) is $|z_1 - z_2|$. This is clear from Fig. 4, since $|z_1 - z_2|$ is the length of the vector representing the number $z_1 - z_2 = z_1 + (-z_2)$ and by translating the radius vector $z_1 - z_2$, one can interpret $z_1 - z_2$ as the directed line segment from the point (x_2, y_2) to the point (x_1, y_1).

Alternatively, it follows from the expression

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$$

and from definition

$$|z_1 - z_2| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}.$$

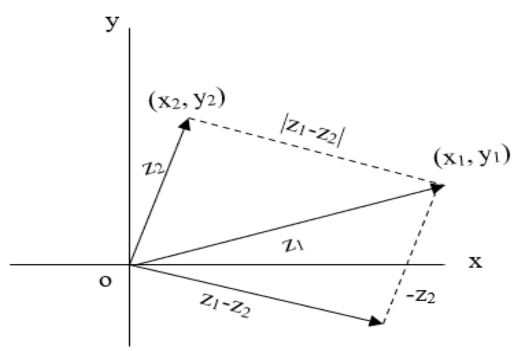


Figure 4

1.1.4 Complex Conjugates: The complex conjugate, or simply the conjugate, of a complex number $z = x + iy$ is defined as the complex number x − iy and is denoted by $\bar{z}$, that is, $\bar{z} = x - iy$. The number $\bar{z}$ is represented by the point $(x, -y)$, which is the reflection in the real axis of the point (x, y) representing z (Fig. 5). Also $\bar{\bar{z}} = z$ and $|\bar{z}| = |z|$ for all z.

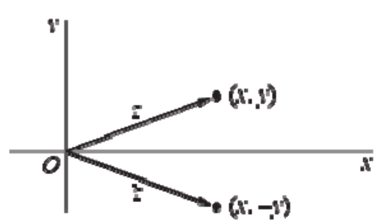


Figure 5

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$\overline{z_1+z_2} = (x_1+x_2) - i(y_1+y_2) = (x_1 - iy_1) + (x_2 - iy_2).$$

So, the conjugate of the sum of the conjugates is

$$\overline{z_1+z_2} = \overline{z_1} + \overline{z_2}.$$

In the same manner, it is easy to show that

$$\overline{z_1-z_2} = \overline{z_1} - \overline{z_2}$$

$$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$$

and

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}} \qquad (z_2 \neq 0).$$

An important identity relating the conjugate of a complex number $z = x + iy$ to its modulus is $z\overline{z} = |z|^2$, where each side is equal to $x^2 + y^2$.

1.1.5 Exponential form: Let r and θ be polar coordinates of the point (x, y) that corresponds to a nonzero complex number $z = x + iy$. Since $x = r\cos\theta$ and $y = r\sin\theta$, the number z can be written in polar form as $z = r(\cos\theta + i\sin\theta)$. If $z = 0$, the coordinate θ is undefined; and so it is understood that $z \neq 0$, whenever polar coordinates are used.

In complex analysis, the real number *'r'* is not allowed to be negative and is the length of the radius vector for z, that is, $r = |z| = \sqrt{x^2 + y^2}$. The real number θ represents the angle, measured in radians, that z makes with the positive real axis when z is interpreted as a radius vector (Fig. 6). As in calculus, θ has an infinite number of possible values, including negative ones, that differ by integral multiples of 2π. Those values can be determined from the equation $\tan\theta = y/x$, where the quadrant containing the point corresponding to z must be specified. Each value of θ is called an argument of z and the set of all such values is denoted by $arg\ z$. The principal value of $arg\ z$, denoted by $Arg\ z$, is that unique value Θ such that $-\pi < \Theta \leq \pi$. Evidently, then,

$$arg\ z = Arg z + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots). \qquad (*)$$

Also, when z is a negative real number, $Arg\ z$ has value π, not $-\pi$.

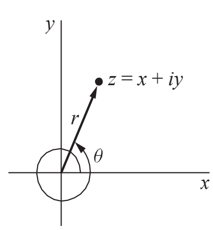


Figure 6

1.1.6 Example: The complex number $-1-i$, which lies in the third quadrant, has principal argument $-3\pi/4$. That is,

$$Arg(-1-i) = -\frac{3\pi}{4}$$

It must be emphasized that because of the restriction $-\pi < \Theta \leq \pi$ of the principal argument Θ, it is not true that

$$Arg(-1-i)=\frac{5\pi}{4}.$$

According to equation (*),

$$arg(-1-i)=-\frac{3\pi}{4}+2n\pi \qquad (n=0,\pm1,\pm2,...)$$

Note that the term *Arg z* on the right hand side of equation (2) can be replaced by any particular value of *arg z* and that one can write, for instance,

$$arg\left(-1-i\right)=-\frac{5\pi}{4}+2n\pi \qquad \left(n=0,\pm1,\pm2,...\right)$$

The symbol $e^{i\theta}$, or $exp(i\theta)$ is defined by means of Euler's formula as $e^{i\theta}=\cos\theta+i\sin\theta$, where θ is to be measured in radians. It enables one to write the polar form more compactly in exponential form as $z=re^{i\theta}$.

1.1.7 Mapping: Let S and T be two non-empty sets in complex plane. If corresponding to every point z of the set S there can be assigned a unique point of w of the set T by means of a rule *'f'*, then we say that *'f'* is a mapping from S to T and we write it as $f: S \to T$.

1.1.8 Functions of Complex Variable: Let S and T be two non-empty sets in complex plane. If corresponding to each value of a complex variable $z = x + iy$ of the set S, there correspond one or more values of another complex variable $w = u + iv$ of the set T, then w is called a function of complex variable z and is denoted by

$$w = f(z) = f(x + iy) = u + iv.$$

If z and w be separated into their real and imaginary parts then the relation $w = f(z)$ becomes $u + iv = f(x + iy)$. From here, it is clear that u and v, in general, depend upon x and y in a certain definite manner so that the function $w = f(z)$ is nothing but the ordered pair of two real functions u and v of two real variables x and y so that we may write $w = u(x, y) + iv(x, y)$.

Functions of (x, y) that depend only on the combination $(x + iy)$ are called functions of a complex variable, with rectangular coordinates x and y. *e.g.* $w = z^2$

$$u + iv = (x + iy)^2 = x^2 - y^2 + 2ixy$$
$$=(x^2 - y^2) + i(2xy).$$

Compare real and imaginary parts, we have

$$u = x^2 - y^2 \text{ and } v = 2xy.$$

Thus, u and v are the functions of the real variables x and y. Therefore, $w = f(z) = u(x, y) + iv(x, y)$.

1.1.9 Single valued Function: If to each value of z there corresponds one and only one values of w, then w is called a single valued function of z. e.g. $w = 1/z\ (z \neq 0)$ is a single valued function.

1.1.10 Multivalued Function: If to each value of z there corresponds more than one value of w, then w is called a multivalued function of z. e.g. $w^2 = z$ is multivalued function of z, because w assumes two values for each value of z except at $z = 0$.

1.1.11 Bounded Function: A function $f(z)$ is said to be bounded in domain D if there exist $k > 0$ such that $|f(z)| \le k$, for all z in D. If $f(z)$ is continuous in a bounded closed region D, then it is bounded in domain D.

1.1.12 Limit of a Complex Function: Let D be the domain of the function in complex plane where functions $f(z)$ is defined. Let z_0 be any point in D, the function $f(z)$ is said to converge or tend to the limit l as z tend to z_0 in any manner in D, if for any $\varepsilon > 0$ however small there exist $\delta > 0$ depending upon ε and z_0 such that

$$|f(z)-l| < \varepsilon \text{, whenever } |z-z_0| < \delta .$$

Where z is other than z_0 and we can also write

$$\lim_{z\to z_0} f(z) = l \text{ or } f(z) \to l \text{ as } z \to z_0 .$$

1.1.13 Continuity of a Complex Function: A complex function $w = f(z)$ defined in the bounded closed domain D, is said to be **continuous** at a point $z = z_0$ of D, if given any positive number ε, we can find a positive number δ such that

$$|f(z) - f(z_0)| < \varepsilon, \text{ whenever } |z - z_0| < \delta.$$

We can also write $\lim_{z\to z_0} f(z) = f(z_0)$

This means that for continuity at a point, the limiting value and the functional value at the point have the same value. A function $f(z)$ is continuous in a domain D if it is continuous at every point of D. If a function is not continuous at z_0, then we say that function is discontinuous at z_0 or z_0 is the point of discontinuity.

1.1.14 Remark: If $f(z)$ is continuous at z_0, then it can be easily shown that

$$\lim_{z\to z_0} f(z) = f(z_0)$$

$$\Leftrightarrow \lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0, \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0,$$

where $f(z_0) = u_0 + iv_0$. This adds the information that the component functions $u(x, y)$ and $v(x, y)$ are also continuous at $z_0 = (x_0, y_0)$.

1.1.15 Remark: If the function $f(z)$ is continuous, so are $|f(z)|$, $f(\bar{z})$ and $\overline{f(z)}$. Suppose that f and g are continuous functions at the point z_0, then the following functions are also continuous at z_0.

(i) $f(z) + g(z)$ (ii) $f(z) - g(z)$

(iii) $f(z).g(z)$ (iv) $f(z)/g(z)$, provided $g(z_0)\neq 0$.

1.1.16 Example: Show that the following functions are continuous everywhere in complex plane

(i) $f(z) = |z|$ (ii) $f(z) = \bar{z}$.

Solution. (i) Here $f(z) = |z|$

Consider $|f(z) - f(z_0)| = ||z|-|z_0||$

$$\leq |z - z_0| < \varepsilon, \text{ whenever } |z - z_0| < \varepsilon = \delta \; [\because ||a|-|b|| \leq |a-b|]$$

Therefore, $f(z)$ is continuous at $z = z_0$.

(ii) Here $f(z) = \bar{z}$

Consider $|f(z) - f(z_0)| = |\bar{z} - \overline{z_0}| = |z - z_0|$ $[\because |\bar{z}| = |z|]$

$$< \varepsilon, \text{ whenever } |z - z_0| < \varepsilon = \delta. \quad [\text{By taking } \varepsilon = \delta]$$

Therefore, $f(z)$ is continuous at $z = z_0$.

1.1.17 Uniform Continuity: A function $f(z)$ is said to be uniformly continuous in domain if for any $\varepsilon > 0$, however small there exist a $\delta > 0$ depending upon ε and independent of z_0 belonging to D such that

$$|f(z) - f(z_0)| < \varepsilon \text{ whenever } |z - z_0| < \delta, \text{ where } z \text{ is different from } z_0.$$

1.1.18 Differentiability of a Complex Function: Let $f(z)$ be a single valued complex functions defined in a domain D of the complex plane. We say that $f(z)$ is differentiable at a point $z_0 \in$ D if the increment ratio $\frac{f(z) - f(z_0)}{z - z_0}$ tends to a finite limit as z tends to z_0 in any manner, provided that z always remains a point of D. This finite limit, if exists, is termed as the differential coefficient or the derivative of $f(z)$ at $z = z_0$ and is denoted by $f'(z_0)$. Thus,

$$\lim_{z \to z_0} \frac{f(z_0 + h) - f(z_0)}{h} = f'(z_0)$$

In other words, let $f(z)$ be a function defined in some domain D containing the neighbourhood of a point z_0. Then $f(z)$ is said to be differentiable at $z = z_0$ if the increment ratio $\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$ tends to a unique limit (finite) as $\Delta z \to 0$ along any path of the domain D, and this unique limit is called the derivative of $f(z)$ at $z = z_0$ and is denoted by $f'(z_0)$. Thus,

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Moreover, we drop the suffix from z_0 and usually write

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

For $w = f(z)$, let $w + \Delta w = f(z + \Delta z)$. Then,

$$\frac{dw}{dz} = f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}.$$

If we get different values of this limit as $\Delta z \to 0$ from different points of D, we say that the derivative of *f(z)* at any $z \in$ D does not exist and the function *f(z)* is said to be non-differentiable at $z \in$ D.

We can also put our definition of differentiability more precisely as follows:

The function *f(z)*, defined and one valued in a domain D of the complex plane, is said to be differentiable at a point $z_0 \in$ D if there exists a definite number l (say) with the property that given any positive number ε, we can find a positive number δ (depending on ε) such that

$$\left|\frac{f(z)-f(z_0)}{z-z_0}-l\right|<\varepsilon$$

for every $z \in$ D in the neighbourhood of z_0 defined by $|z - z_0| < \delta$. When this is the case, we call l the derivative or differential coefficient of *f(z)* at z_0 and denote it by $f'(z_0)$.

If f is differentiable at each point of D, we say that f is differentiable on D. We observe that if f is differentiable on D, then $f'(z)$ defines a function f': D→ℂ. If f is continuous, then we say that f is continuously differentiable. If f' is differentiable, then f is said to be twice differentiable. Continuing in this manner, a differentiable function such that each successive derivative is again differentiable, is called infinitely differentiable. It is immediate that the derivative of a constant function is zero.

1.1.19 Theorem: Every differentiable function is continuous.

Proof: If f is differentiable at a point z_0 in D, then

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} \text{ exists finitely.}$$

Consider

$$\lim_{z \to z_0}\left[f(z)-f(z_0)\right] = \lim_{z \to z_0}\left[\frac{f(z)-f(z_0)}{z-z_0}\right]\lim_{z \to z_0}(z-z_0)$$

$$= f'(z_0).0 = 0$$

i.e. $\lim_{z \to z_0} f(z) = f(z_0)$. Hence, $f(z)$ is continuous at z_0.

1.1.20 Remark: Converse of the above theorem is not necessarily true.

1.1.21 Example: Consider the function $f(z)=|z|^2$. This function is continuous in all finite region of *z*-plane but nowhere differentiable except at origin.

Solution: Here $f(z)=|z|^2 = x^2+y^2$

$\therefore u(x,y)=x^2+y^2$ and $v(x,y)=0$. Clearly, *f(z)* is continuous everywhere because of the continuity of *u(x, y)* and *v(x, y)*. Consider,

$$f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}, z_0\neq(0,0)$$

$$=\lim_{z\to z_0}\frac{|z|^2-|z_0|^2}{z-z_0}=\lim_{z\to z_0}\frac{z.\bar{z}-z_0\overline{z_0}}{z-z_0} \qquad \left[\because |z|^2=z.\bar{z}\right]$$

In numerator, adding and subtracting $\bar{z}.z_0$

$$f'(z_0)=\lim_{z\to z_0}\frac{z.\bar{z}-\bar{z}.z_0+\bar{z}.z_0-z_0\overline{z_0}}{z-z_0}$$

$$=\lim_{z\to z_0}\frac{\bar{z}(z-z_0)+z_0(\bar{z}-\overline{z_0})}{z-z_0}$$

$$=\lim_{z\to z_0}\bar{z}+\frac{z_0(\bar{z}-\overline{z_0})}{z-z_0}. \tag{1}$$

If we put $z-z_0=r(\cos\theta+i\sin\theta)$ then $\bar{z}-\overline{z_0}=\overline{z-z_0}=r(\cos\theta-i\sin\theta)$.

So that $\dfrac{\bar{z}-\overline{z_0}}{z-z_0}=\dfrac{r(\cos\theta-i\sin\theta)}{r(\cos\theta+i\sin\theta)}=\dfrac{e^{-i\theta}}{e^{i\theta}}=e^{-2i\theta}=\cos 2\theta-i\sin 2\theta$.

Thus, equation (1) becomes

$$f'(z_0)=\lim_{z\to z_0}\left[\bar{z}+z_0(\cos 2\theta-i\sin 2\theta)\right], \text{ where } \theta=\arg(z-z_0) . \tag{2}$$

This last expression in (2) does not tend to unique limit as $z\to z_0$ in any manner. Since this limit depends upon θ. Therefore function is not differentiable at non-zero values of *z*. At z_0 = (0, 0) the expression on the R.H.S. of (2) becomes $\bar{z}$ which of course tend to zero with *z* and thus function is differentiable at zero (origin).

1.1.22 Example: Consider the function $f(z)=\bar{z}$. This function is obviously continuous but does not possess derivative.

Solution: Since, by definition

$$f'(z)=\lim_{h\to 0}\frac{\overline{z+h}-\bar{z}}{h}=\lim_{h\to 0}\frac{\bar{h}}{h}$$

If we write $h=r\,e^{i\theta}$, then

$$f'(z)=\lim_{h\to 0}e^{-2i\theta}$$

So, if $h\to 0$ along the positive real axis ($\theta=0$), then $f'(z)=1$ and if $h\to 0$ *along* the positive imaginary axis $(\theta=\pi/2)$, then $f'(z)=-1$. Hence $f'(z)$ is not unique and it depends on how *h* approaches zero. Thus, we find the surprising result that the function $f(z)=\bar{z}$ is not differentiable anywhere, even though it is continuous everywhere.

1.1.23 Example: Find $f'(z)$, where $f(z) = z^3 + 2z^2 + i$.

Solution. Consider $\dfrac{f(z+\Delta z)-f(z)}{\Delta z} = \dfrac{[(z+\Delta z)^3 + 2(z+\Delta z)^2 + \mathrm{i}] - [\mathrm{z}^3 + 2\,\mathrm{z}^2 + \mathrm{i}]}{\Delta z}$

$$= 3z^2 + 4z + 3z\Delta z + 2\Delta z + (\Delta z)^2$$

Taking limit $\Delta z \to 0$, we get

$$f'(z) = 3z^2 + 4z$$

By ε-δ method, for given $\varepsilon > 0$, we must find $\delta > 0$ such that

$$\left|\frac{f(z+\Delta z)-f(z)}{\Delta z} - f'(z)\right| < \varepsilon \text{ whenever } |\Delta z| < \delta .$$

Consider

$$\left|\frac{f(z+\Delta z)-f(z)}{\Delta z} - f'(z)\right| = \left|3z^2 + 4z + 3z\Delta z + 2\Delta z + (\Delta z)^2 - (3z^2 + 4z)\right|$$

$$= |3z + 2 + \Delta z||\Delta z|$$

Restricting $|\Delta z| < 1$, we observe that

$$|3z + 2 + \Delta z| \le 3|z| + 2 + |\Delta z|$$

$$< 3(1 + |z|).$$

If we choose $\delta = \min\left\{1, \dfrac{\varepsilon}{3(1+|z|)}\right\}$, then $f(z)$ is derivable at $z \in$ D.

1.2 Analytic Function: Let D be an open set in $\mathbb{C}$. A function f : D $\to \mathbb{C}$ is analytic (holomorphic) in D if $f(z)$ is differentiable at each point of D. Here, it is important to stress that the open set D is a part of the definition.

Equivalently, a function $f(z)$ is said to be analytic at $z = z_0$ if $f(z)$ is differentiable at every point of some neighbourhood of z_0 i.e. $f(z)$ is said to be analytic at $z = z_0$ if there exist a neighbourhood $|z-z_0| < \delta$ at every point of which $f'(z)$ exists. We observe that $f(z) = |z - z_0|^2$ is differentiable at $z = z_0$ but it is not analytic at $z = z_0$ because there does not exist a neighbourhood of 'z_0' in which $|z - z_0|^2$ is differentiable at each point of the neighbourhood.

If in a domain D of the complex plane, $f(z)$ is analytic throughout, we sometimes say that $f(z)$ is regular in D to emphasize that every point of D is a point at which $f(z)$ is analytic. Further, if $f(z)$ is analytic at each point of the entire finite plane, then $f(z)$ is called an **entire function(regular function)**. A point where the function fails to be analytic, is called a **singular point** or singularity of the function.

The set (class) of functions holomorphic in D is denoted by H(D). The usual differentiation rules apply for analytic functions. Thus, if $f, g \in$ H(D), then $f + g \in$ H(D) and $fg \in$ H(D), so that H(D) is a ring. Further, superpositions of analytic functions are analytic, chain rule of differentiation applies. Thus, if f and g are analytic on D and D_1 respectively and $f(D) \subset D_1$, then gof is analytic on D and $(gof)'(z) = g'(f(z))f'(z)$ for all z in D.

1.2.1 Example: Show that $f(z)=\bar{z}$ is not analytic anywhere.

Solution: Here, $f(z)=\bar{z}$.

By definition, $f'(z)=\lim\limits_{\delta z\to 0}\dfrac{f(z+\delta z)-f(z)}{\delta z}$

$$=\lim_{\delta z\to 0}\frac{\overline{z+\delta z}-\bar{z}}{\delta z}$$

$$=\lim_{\delta x\to 0,\delta y\to 0}\frac{\overline{x+iy+\delta x+i\delta y}-\overline{x+iy}}{\delta x+i\delta y}$$

$$=\lim_{\delta x\to 0,\delta y\to 0}\frac{x-iy+\delta x-i\delta y-x+iy}{\delta x+i\delta y}$$

$$=\lim_{\delta x\to 0,\delta y\to 0}\frac{\delta x-i\delta y}{\delta x+i\delta y}$$

If $\delta y=0$ then $f'(z)=\lim\limits_{\delta x\to 0}\dfrac{\delta x}{\delta x}=1$.

If $\delta x=0$ then $f'(z)=\lim\limits_{\delta y\to 0}\left(\dfrac{-i\delta y}{i\delta y}\right)=-1$.

Thus, $f'(z)$ does not exist i.e. *f (z)* is not differentiable. Hence, *f (z)* is not analytic at anywhere.

1.2.2 Exercise: Examine the nature of the function *f(z)* in a region including origin.

$$f(z)=\begin{cases}\dfrac{x^2y^5(x+iy)}{x^4+y^{10}}, & z\neq 0\\ 0 & , z=0\end{cases}$$

Solution: Here $f(z)=\dfrac{x^3y^5+ix^2y^6}{x^4+y^{10}}=\dfrac{x^3y^5}{x^4+y^{10}}+i\dfrac{x^2y^6}{x^4+y^{10}}$

$$\Rightarrow \qquad u=\frac{x^3y^5}{x^4+y^{10}} \text{ and } v=\frac{x^2y^6}{x^4+y^{10}}$$

By definition,

$$f'(0)=\lim_{z\to 0}\frac{f(z)-f(0)}{z}=\lim_{z\to 0}\frac{f(z)}{z}$$

$$=\lim_{x\to 0}\frac{x^2y^5(x+iy)}{x^4+y^{10}}\cdot\frac{1}{(x+iy)}=\lim_{x\to 0}\frac{x^2y^5}{x^4+y^{10}}$$

Let $z\to 0$ along the path $y=x$, then

$$f'(0) = \lim_{x\to 0}\frac{x^2y^5}{x^4+y^{10}} = \lim_{x\to 0}\frac{x^7}{x^4\left(1+x^6\right)} = \lim_{x\to 0}\frac{x^3}{1+x^6} = 0$$

Again, let $z \to 0$ along the path $y^5 = x^2$ then

$$f'(0) = \lim_{x\to 0}\frac{x^4}{x^4+x^4} = \lim_{x\to 0}\frac{1}{2} = \frac{1}{2}$$

Thus, $f'(0)$ is not unique, so $f'(0)$ does not exist. Hence, *f(z)* is not analytic at $z = 0$. Consequently, *f(z)* is not analytic in a region including origin.

1.2.3 Remark: The theory of analytic functions cannot be considered as a simple generalization of calculus. To point out how vastly different the two subjects are, we shall show that every analytic function is infinitely differentiable and also has a power series expansion about each point of its domain. These results have no analogue in the theory of functions of real variables. Further, in the complex variable case, there are an infinity of directions in which a variable z can approach a point z_0, at which differentiability is considered. In the real case, however, there are only two avenues of approach (e.g. continuity of a function in real case, can be discussed in terms of left and right continuity).

Thus, we notice that the statement that a function of a complex variable has a derivative is stronger than the same statement about a function of a real variable.

1.2.4 Cauchy-Riemann Equations: Now we come to the earlier mentioned compatibility relationship between the real and imaginary parts of a complex function which are necessarily satisfied if the function is differentiable. These relations are known as Cauchy-Riemann equations (CR equations). We have seen that every complex function can be expressed as $f(z)=u(x, y)+iv(x, y)$, where $u(x, y) \equiv u$ and $v(x, y) \equiv v$ are real functions of two real variables x and y. We shall denote the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x \partial y}$ by $u_x, u_y, u_{xx}, u_{yy}, u_{xy}$ respectively.

1.2.5 Theorem: (Necessary condition for *f(z)* to be analytic). If a function $f(z) = u(x, y) + iv(x, y)$ is differentiable at any point $z_0 = x_0 + iy_0$ in a domain D, then the four partial derivatives u_x, u_y, v_x, v_y exist and satisfy the equations $u_x = v_y$ and $u_y = -v_x$.

Proof: Since $f(z) = u(x, y) + iv(x, y)$ is differentiable at any point z_0 in D,

$$f'(z_0) = \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0},$$

exists finitely and uniquely as $z \to z_0$ in any manner.

Suppose $z \to z_0$ in such a manner that $z \to z_0$ is purely real i.e. along x-axis. So, let $z_0 = x_0 + iy_0$ and $z = x + iy_0$ such that $z - z_0 = x + iy_0 - x_0 - iy_0 = x - x_0 =$ real

Also if $z \to z_0$ then $x \to x_0$.

Now $$f'(z_0) = \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} \qquad (1)$$

$$= \lim_{z \to z_0} \frac{f(x+iy_0) - f(x_0+iy_0)}{x-x_0}$$

$$= \lim_{x \to x_0} \frac{[u(x,y_0)+iv(x,y_0)] - [u(x_0,y_0)+iv(x_0,y_0)]}{x-x_0}$$

$$= \lim_{x \to x_0} \frac{u(x,y_0)-u(x_0,y_0)}{x-x_0} + i \lim_{x \to x_0} \frac{v(x,y_0)-v(x_0,y_0)}{x-x_0} \tag{2}$$

If we assume that the function is differentiable then this expression must tend to a unique limit as $x \to x_0$. Thus, real and imaginary parts of the expression must tend to unique limits. This is equivalent to the statement that partial differential coefficient $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ exists at the point *(x_0, y_0)* or z_0. Hence, we get

$$f'(z_0) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \tag{3}$$

Similarly, let us take the mode of tending of z to z_0 such that $z - z_0$ is purely imaginary (i.e. along y-axis). By taking $z_0 = x_0 + iy_0$ and $z = x_0 + iy$. Therefore, $z - z_0 = i(y - y_0)$ = purely imaginary.

Also if $z \to z_0$ then $y \to y_0$. Now,

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{y \to y_0} \frac{[u(x_0,y)+iv(x_0,y)] - [u(x_0,y_0)+iv(x_0,y_0)]}{i(y-y_0)}$$

$$= \lim_{y \to y_0} \left[\frac{u(x_0,y)-u(x_0,y_0)}{i(y-y_0)} + \frac{i[v(x_0,y)-v(x_0,y_0)]}{i(y-y_0)} \right]$$

$$= \lim_{y \to y_0} \left[\frac{v(x_0,y)-v(x_0,y_0)}{(y-y_0)} \right] - i \lim_{y \to y_0} \left[\frac{u(x_0,y)-u(x_0,y_0)}{(y-y_0)} \right] \tag{4}$$

Proceeding as above we conclude that partial derivatives $\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}$ exists at z_0 or *(x_0, y_0)* and thus we get

$$f'(z_0) = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \tag{5}$$

Since *f(z)* is given to be analytic therefore $f'(z_0)$ is unique derivative of *f(z)* at $z = z_0$. So, comparing (3) and (5) and equating real and imaginary parts, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{6}$$

i.e. $u_x = v_y$ and $v_x = -u_y$ or $u_y = -v_x$.

The equations given by (6) are known as C-R equations.

1.2.6 Remark: We have $f(z) = u + iv$ which gives

$$\frac{\partial f}{\partial x}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x},\frac{\partial f}{\partial y}=\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}$$

From these two results, C-R equations, in complex form, can be put as

$$\frac{\partial f}{\partial x}=\frac{1}{i}\frac{\partial f}{\partial y}$$

(ii) We note that unless the differential equations (3) i.e. C-R equations are satisfied, $f(z) = u + iv$ cannot be differentiable at any point even if the four first order partial derivatives exist.

For example, let us take

$$f(z) = Re\, z = x,\ z = x + iy.$$

Then $$\frac{\partial u}{\partial x}=1,\frac{\partial u}{\partial y}=0,\frac{\partial v}{\partial x}=0,\frac{\partial v}{\partial y}=0$$

Thus, although the partial derivatives exist everywhere, C-R equations are not satisfied at any point of the complex plane.

Hence, the function $f(z) = Re\, z$ is not differentiable at any point.

1.2.7 Remark: The conditions of the theorem (1.2.5) are not sufficient. Actually, C-R equations are useful for proving non-differentiability. They are not, on their own, a sufficient condition for differentiability.

1.2.8 Example: We consider the function

$$f(z)=\begin{cases}\left(\bar{z}\right)^2/z, z\neq 0\\ 0 \qquad\quad ,z=0\end{cases}, \qquad z=x+iy$$

and show that $f(z)$ is not differentiable at the origin, although C-R equations are satisfied at that point.

Solution: By definition, we have

$$f'(0)=\lim_{z\to 0}\frac{f(z)-f(0)}{z}=\lim_{z\to 0}\frac{\left(\bar{z}\right)^2}{z^2}$$

$$=\lim_{(x,y)\to(0,0)}\left(\frac{x-iy}{x+iy}\right)^2$$

$$=\begin{cases}1\, if\ z\to 0\, along\ real\ axis\\ 1\, if\ z\to 0\, along\ imaginary\ axis\\ -1\, if\ z\to 0\, along\ the\ line\ y=x\end{cases}$$

Thus, $f'(0)$ is not unique and hence $f(z)$ is not differentiable at the origin.

Now, to verify C-R equations, we have $f(0)= 0$ implies $u(0,0)= 0$, $v(0,0)= 0$.

Also $\quad f(z) = \dfrac{(\bar{z})^2}{z} = \dfrac{(\bar{z})^3}{z\bar{z}} = \dfrac{(x-iy)^3}{x^2+y^2}$

From here, $u(x,y) = \dfrac{x^3-3xy^2}{x^2+y^2}, v(x,y) = \dfrac{y^3-3x^2y}{x^2+y^2}$

Therefore, at $(0,0)$

$u_x = 1, u_y = 0, v_x = 0, v_y = 1$ Thus, C-R equations are satisfied at the origin.

1.2.9 Example: Show that the function $f(z) = \sqrt{|xy|}, z = x+iy$ is not analytic at the origin, Although, C-R equations are satisfied at that point.

Solution: Here, $\quad f(z) = \sqrt{|xy|} = u(x,y) + iv(x,y)$

So that $u(x,y) = \sqrt{|xy|}$ and $v(x,y) = 0$

Now, at $z = (0, 0)$

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x-0} = \lim_{x\to 0} \frac{0}{x} = 0.$$

Similarly, $\dfrac{\partial u}{\partial y} = \lim\limits_{y\to 0} \dfrac{u(0,y)-u(0,0)}{y-0} = \lim\limits_{y\to 0} \dfrac{0}{y} = 0.$

And $\quad \dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y} = 0.$

i.e. $u_x = v_y$ and $u_y = -v_x$.

Hence, all the four partial derivatives exist and satisfy C-R equations.

$$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{z\to 0} \frac{\sqrt{|xy|}}{x+iy} = \lim_{x\to 0} \frac{\sqrt{|xy|}}{x+iy}$$

Let $z \to 0$, along the path $y = mx$. Then

$$f'(0) = \lim_{z\to 0} \frac{x\sqrt{m}}{x(1+im)} = \frac{\sqrt{m}}{1+im}.$$

Since value of this limit depends upon m i.e. $f'(0)$ is not unique. It means that $f'(0)$ doesn't exist and so $f(z)$ is not analytic at $z = 0$.

To make C-R equations as sufficient an additional condition of continuity on partial derivatives is imposed.

1.2.10 Theorem (Sufficient condition for *f(z)* to be analytic): Suppose that $f(z) = u(x, y) + iv(x, y)$ for $z = x + iy$ is analytic in a region D, if the four partial derivatives u_x, u_y, v_x, v_y exists, continuous and satisfy Cauchy-Riemann equations at each point of D.

Proof. Consider the point $z = (x, y)$ in a region D. Let $(x+\delta x, y+\delta y)$ be the point in the neighbourhood of point *(x, y)*. Let $w = f(z) = u(x,y) + iv(x,y), z = x+iy$.

Now, $u = u(x,y)$ (1)

and $u + \delta u = u(x+\delta x, y+\delta y)$. (2)

$$\Rightarrow \delta u = u(x+\delta x, y+\delta y) - u(x,y)$$

Similarly, $\delta v = v(x+\delta x, y+\delta y) - v(x,y)$

Since u_x, u_y, v_x and v_y are continuous in a region G and we have by the **mean value theorem** for functions of two variables.

$$\delta u = \left(\frac{\partial u}{\partial x} + \in\right)\delta x + \left(\frac{\partial u}{\partial y} + \eta\right)\delta y, \tag{3}$$

where $\in$ and η are small and tends to zero as δx and δy tends to zero.

Similarly, applying mean value theorem for *v(x, y),* we get

$$\delta v = \left(\frac{\partial v}{\partial x} + \in'\right)\delta x + \left(\frac{\partial v}{\partial y} + \eta'\right)\delta y \tag{4}$$

where $\in'$ and η' are small and tends to zero as δx and δy tends to zero. Now, using C-R equations i.e. $u_x = v_y$ and $u_y = -v_x$, we have

$$\delta u + i\delta v = \left(\frac{\partial u}{\partial x} + \in\right)\delta x + \left(\frac{\partial u}{\partial y} + \eta\right)\delta y + i\left(\frac{\partial v}{\partial x} + \in'\right)\delta x + i\left(\frac{\partial v}{\partial y} + \eta'\right)\delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y + (\in + i\in')\delta x + (\eta + i\eta')\delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\delta x + i\delta y) + (\in + i\in')\delta x + (\eta + i\eta')\delta y \tag{5}$$

Now, by definition

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{\delta u + i\delta v}{\delta x + i\delta y}$$

$$f'(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \lim_{\delta z \to 0} \frac{(\in + i\in')\delta x + (\eta + i\eta')\delta y}{\delta x + i\delta y} \tag{6}$$

Now

$$\left|\frac{(\in + i\in')\delta x + (\eta + i\eta')\delta y}{\delta x + i\delta y}\right| \leq \frac{(|\in| + |\in'|)|\delta x| + (|\eta| + |\eta'|)|\delta y|}{|\delta x + i\delta y|}$$

$$\leq \frac{(|\in| + |\in'|)|\delta x|}{|\delta x + i\delta y|} + \frac{(|\eta| + |\eta'|)|\delta y|}{|\delta x + i\delta y|}$$

$$\leq |\in| + |\in'| + |\eta| + |\eta'|, \text{ using } |\delta x| \leq |\delta x + i\delta y|, |\delta y| \leq |\delta x + i\delta y|. \qquad (7)$$

But the expression on R.H.S. of (7) tends to zero as δx and δy tend to zero. Finally from (6), we get

$$f'(z) = \frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}.$$

Which is finite and definite at every point of D. Hence, $f(z)$ is analytic at every point of D.

1.2.11 Theorem: Let u and v be real-valued functions defined on a region D and suppose that u and v have continuous first order partial derivatives. Then f : D → $\mathbb{C}$ defined by $f(z) = u(x, y) + iv(x, y)$ is analytic iff u and v satisfy the Cauchy-Riemann equations.

1.2.12 C-R Equations in Polar Form: If $f(z) = u + iv$ be an analytic function and $z = re^{i\theta}$, where u, v, r, θ are real, then the C-R equations are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad and \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}.$$

Proof: In polar co-ordinates (r, θ), $x = r\cos\theta$, $y = r\sin\theta$. Therefore,

$$r = \sqrt{x^2 + y^2} \quad , \theta = \tan^{-1}\frac{y}{x}$$

Now,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$= \frac{\partial u}{\partial r}\left(\frac{x}{\sqrt{x^2 + y^2}}\right) + \frac{\partial u}{\partial \theta}\left(-\frac{y}{x^2 + y^2}\right)$$

$$= \frac{\partial u}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta \qquad (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$= \frac{\partial u}{\partial r}\left(\frac{y}{\sqrt{x^2 + y^2}}\right) + \frac{\partial u}{\partial \theta}\left(-\frac{x}{x^2 + y^2}\right)$$

$$= \frac{\partial u}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial u}{\partial \theta}\cos\theta \qquad (2)$$

Similarly, $$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial v}{\partial \theta}\sin\theta \qquad (3)$$

and $$\frac{\partial v}{\partial y}=\frac{\partial v}{\partial r}\sin\theta-\frac{1}{r}\frac{\partial v}{\partial \theta}\cos\theta \tag{4}$$

Using CR equations $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ with (1) and (4), $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ with (2) and (3), we get

$$\left(\frac{\partial u}{\partial r}-\frac{1}{r}\frac{\partial v}{\partial \theta}\right)\cos\theta-\left(\frac{\partial v}{\partial r}+\frac{1}{r}\frac{\partial u}{\partial \theta}\right)\sin\theta=0 \tag{5}$$

$$\left(\frac{\partial u}{\partial r}-\frac{1}{r}\frac{\partial v}{\partial \theta}\right)\sin\theta+\left(\frac{\partial v}{\partial r}+\frac{1}{r}\frac{\partial u}{\partial \theta}\right)\cos\theta=0 \tag{6}$$

Multiplying (5) by *cos θ* and (6) by *sin θ* and then adding, we find

$$\frac{\partial u}{\partial r}-\frac{1}{r}\frac{\partial v}{\partial \theta}=0$$

i.e $$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta} \tag{7}$$

Again, multiplying (5) by *sin θ* and (6) by *cos θ* and then subtracting, we have

$$\frac{\partial v}{\partial r}+\frac{1}{r}\frac{\partial u}{\partial \theta}=0$$

i.e $$\frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta} \tag{8}$$

Equations (7) and (8) are the required C-R equations in polar co-ordinates.

1.2.13 Remark: We can express $f'(z)$ in polar co-ordinates as

$$\begin{aligned}
f'(z)&=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\\
&=\frac{\partial u}{\partial r}\cos\theta-\frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta+i\frac{\partial v}{\partial r}\cos\theta-i\frac{1}{r}\frac{\partial v}{\partial \theta}\sin\theta\\
&=\left(\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right)\cos\theta-\left(\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right)i\sin\theta\\
&=\left(\cos\theta-i\sin\theta\right)\left(\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right)\\
&=e^{-i\theta}\left(\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right)
\end{aligned}$$

i.e $\frac{dw}{dz}=e^{-i\theta}\frac{\partial w}{\partial r}$

similarly, we get $\frac{dw}{dz}=-\frac{1}{r}e^{-i\theta}\frac{\partial w}{\partial \theta}$.

1.2.14 Definition: A function $w = f(z) = u + iv$ ceases to be analytic whenever $\frac{dz}{dw} = 0$ i.e. $\frac{dw}{dz} = \infty$.

1.2.15 Exercise: For what value of z, the function w defined by $z = e^{-v}(\cos u + i\sin u)$ ceases to be analytic.

Solution: Here $z = e^{-v}(\cos u + i\sin u)$

$$z = e^{-v}e^{iu} = e^{iu-v} = e^{i(u+iv)} = e^{iw}$$

$$\therefore \quad z = e^{iw}$$

$$\Rightarrow \log z = iw$$

Differentiating, we have

$$\frac{1}{z} = \frac{idw}{dz} \Rightarrow \frac{dw}{dz} = \frac{1}{iz}.$$

From here, w is not analytic only when $\frac{dz}{dw} = 0$ i.e. $iz = 0$ or $z = 0$. So function ceases to be analytic at the origin.

1.2.16 Theorem: A real function of a complex variables either has derivative zero or the derivative does not exist.

Proof. Suppose that $f(z)$ is a real function of complex variable whose derivative exists at z_0. Then, by definitions

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

Let $h = h_1 + ih_2$. If we take the limit $h \to 0$ along the real axis, $h = h_1 \to 0$, then $f'(z_0)$ is real (since f is real). If we take the limit $h \to 0$ along the imaginary axis, $h = ih_2 \to 0$, then $f'(z_0)$ becomes purely imaginary number, where f is real. So we must have $f'(z_0) = 0$. Further, in this case we also observe that if $f(z)$ is analytic then, using C-R equations, we conclude that $f(z)$ is a constant function.

1.2.17 Example: Show that the function $f(0) = 0$,

$$f(z) = u + iv = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2}$$
$$= \frac{x^3 - y^3}{x^2 + y^2} + i\frac{x^3 + y^3}{x^2 + y^2}$$

is continuous and that the C-R equations are satisfied at the origin, yet $f'(0)$ does not exist.

Solution. We have

$$u = \frac{x^3 - y^3}{x^2 + y^2}, v = \frac{x^3 + y^3}{x^2 + y^2}$$

When $z \neq 0$, u and v are rational functions of x and y with non-zero denominators. It follows that they are continuous when $z \neq 0$. To test the continuity at $z = 0$, we change to polars and get

$$u = r\left(cos^3\theta - sin^3\theta\right),\ v = r\left(cos^3\theta + sin^3\theta\right)$$

each of which tends to zero as $r \rightarrow 0$, whatever value θ may have. Now, the actual values of u and v at origin are zero since $f(0) = 0$. So the actual and the limiting values of u and v at the origin are equal so they are continuous. Hence, $f(z)$ is continuous function for all values of z. Now, at the origin

$$\frac{\partial u}{\partial x} = \lim_{x\to 0}\frac{u(x,0)-(0,0)}{x}$$
$$= \lim_{x\to 0}\frac{x^3/x^2}{x} = 1$$

Similarly, $\frac{\partial u}{\partial y} = -1, \frac{\partial v}{\partial x} = 1, \frac{\partial v}{\partial y} = 1$.

Hence, C-R equations are satisfied at the origin.

Again, $f'(z) = \lim_{z\to 0}\frac{f(z)-f(0)}{z}$

$$= \lim_{z\to 0}\frac{\left(x^3-y^3\right)+i\left(x^3+y^3\right)}{x^2+y^2}.\frac{1}{x+iy}$$

If we let $z \rightarrow 0$ along real axis ($y = 0$), then $f'(0) = 1 + i$.

If $z\rightarrow 0$ along $y = x$, then $f'(0) = \frac{i}{1+i}$

Thus $f'(0)$ is not unique and hence $f(z)$ is not differentiable at the origin. Similar conclusion (as for example 1.2.17) holds for the following two functions

$$(i) f(z) = u + iv = \begin{cases} \frac{\mathrm{I}_m\left(z^2\right)}{|z|^2}, & z \neq 0 \\ 0 & , z = 0 \end{cases}$$

$$(ii) f(z) = u + iv = \begin{cases} \frac{z^5}{|z|^4}, & z \neq 0 \\ 0 & , z = 0 \end{cases}$$

1.2.18 Milne Thomson Method: By this method we can construct an analytic function, when we know its either real or imaginary part. Since, we have

$$x = \frac{z+\bar{z}}{2}, y = \frac{z-\bar{z}}{2i} \text{ where } z = x + iy.$$

Therefore, $f(z) = u(x,y) + iv(x,y)$

$$= u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) \tag{1}$$

Equation (1) can be regarded as a formal identity in two independent variables z and $\bar{z}$. Now, by setting $x = z$ and $y = 0$ so that $z = \bar{z}$. We have

$$f(z) = u(z,0) + iv(z,0).$$

Now, if *f(z)* is analytic, then we have

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad [\because u_y = -v_x]$$

Hence, if we write

$$\frac{\partial u}{\partial x} = \phi(x,y), \frac{\partial u}{\partial y} = \psi(x,y)$$

Then,

$$f'(z) = \phi(x,y) - i\psi(x,y)$$

$$= \phi(z,0) - i\psi(z,0).$$

Integrating w.r.t. z, we get

$f(z) = \int [\phi(z,0) - i\psi(z,0)]dz + c$, c being a constant.

Thus, we can construct *f(z)* if *u(x,y)* is known.

Similarly, if *v(x, y)* is given, then we have

$$f(z) = \int [\phi_1(z,0) + i\psi_1(z,0)]dz + c_1,$$

where $\phi_1(x,y)$ and $\psi_1(x,y)$ denote $\frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x}$ respectively and c_1 is some arbitrary constant.

1.2.19 Example: Construct the analytic function of which real part is $u(x,y) = e^x(x\cos y - y\sin y)$

Solution Here real part *u(x, y)* is known. By Milne-Thomson's method, we have

$$f(z) = \int [\phi(z,0) - i\psi(z,0)]dz + c \tag{1}$$

where, $\phi(z,0) = \frac{\partial u}{\partial x}(z,0)$ *and* $\psi(z,0) = \frac{\partial u}{\partial y}(z,0)$

Now, $\frac{\partial u}{\partial x} = e^x[x\cos y - y\sin y + \cos y]$

$$\frac{\partial u}{\partial y} = e^x\left[-x\sin y - \sin y - y\cos y\right]$$

$\therefore \phi(z,0) = e^z(1+z)\, and\, \psi(z,0) = 0$

From (1), we obtain

$$f(z) = \int\left[(1+z)e^z - i(0)\right]dz + c \qquad \text{[Integrating by parts]}$$

$$= \int(1+z)e^z dz + c = ze^z + c.$$

1.2.20 Example: Construct an analytic function $f(z)$ for $u(x, y) = 4xy - x^3 + 3xy^2$.

Solution: Since $u(x, y)$ is given. By Milne-Thomson's Method, we have

$$f(z) = \int\left[\phi(z,0) - i\psi(z,0)\right]dz + c \tag{1}$$

where $\phi(z,0) = \frac{\partial u}{\partial x}(z,0)\, and\, \psi(z,0) = \frac{\partial u}{\partial y}(z,0)$

Now, $\frac{\partial u}{\partial x} = 4y - 3x^2 + 3y^2$

$$\frac{\partial u}{\partial y} = 4x + 6xy$$

$$\therefore \phi(z,0) = -3z^2\, and\, \psi(z,0) = 4z$$

From (1), we obtain

$$f(z) = \int\left[-3z^2 - i4z\right]dz + c = -z^3 - 2iz^2 + c\,.$$

1.2.21 Exercise:

(i) Find the analytic function $f(z) = u + iv$ if $u(x,y) = \log\sqrt{x^2+y^2}$.

(ii) Find analytic function whose imaginary part is $v = e^x(x \sin y + y \cos y)$.

(iii)Show that the function u $(x, y) = e^x \cos y$ is harmonic. Determine its harmonic conjugate $v(x, y)$ and the analytic function $f(z) = u + iv$.

(iv)Show that u $(x, y) = e^{-x}(x \sin y - y \cos y)$ is harmonic and find $v(x, y)$ such that $f(z) = u + iv$ is analytic.

1.2.22 Example: Show that the function $f(z) = e^{-z^{-4}}$ ($z \neq 0$) and $f(0) = 0$ is not analytic at $z = 0$, although C-R equations are satisfied at origin.

Solution: Here $f(z) = e^{-z^{-4}} = e^{-\frac{1}{z^4}} = e^{-\frac{1}{(x+iy)^4}}$

$$= e^{-\left[\frac{(x-iy)^4}{(x+iy)^4(x-iy)^4}\right]} = e^{-\left[\frac{(x-iy)^4}{\left(x^2+y^2\right)^4}\right]} = e^{-\frac{(x-iy)^4}{r^8}}$$

$$= e^{-\frac{1}{r^8}\left[x^4+y^4-6x^2y^2-4ix^3y+4ixy^3\right]}$$

$$= e^{-\frac{1}{r^8}\left[x^4+y^4-6x^2y^2\right]}.e^{-\left[-\frac{1}{r^8}.4ixy\left(x^2-y^2\right)\right]}$$

$$= e^{-\frac{1}{r^8}\left[x^4+y^4-6x^2y^2\right]}.e^{\left[\frac{4ixy\left(x^2-y^2\right)}{r^8}\right]}$$

$$= e^{-\frac{1}{r^8}\left[x^4+y^4-6x^2y^2\right]}\left[\cos\frac{4ixy\left(x^2-y^2\right)}{r^8}+i\sin\frac{4ixy\left(x^2-y^2\right)}{r^8}\right]$$

Hence, at origin we have

$$\frac{\partial u}{\partial x} = \lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x-0} = \lim_{x\to 0}\frac{e^{-x^{-4}}}{x} = \lim_{x\to 0}\frac{1}{xe^{\frac{1}{x^4}}}$$

$$= \lim_{x\to 0}\left[\frac{1}{x\left(1+\frac{1}{x^4}+\frac{1}{2x^8}+\ldots\right)}\right] = \lim_{x\to 0}\left[\frac{1}{x+\frac{1}{x^3}+\frac{1}{2x^7}+\ldots}\right] = 0$$

Similarly, $\frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial y} = 0$ and $\frac{\partial v}{\partial x} = 0$

Hence, C-R equations are satisfied.

To show that *f(z)* is not analytic at origin, we have

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} e^{-z^{-4}}$$

Let $z \to 0$ along the path $z = re^{\frac{i\pi}{4}}$ so that $r \to 0$ as $z \to 0$

$$\lim_{z\to 0} f(z) = \lim_{r\to 0} e^{-r^{-4}e^{-i\pi}}$$

$$= \lim_{r\to 0} e^{r^{-4}} = \lim_{r\to 0} e^{1/r^4} = e^{\infty} = \infty$$

It shows that $\lim_{z\to 0} f(z)$ does not exist means *f(z)* is not continuous at *z = 0*. Therefore, *f(z)* is not differentiable at *z = 0*. Hence, *f(z)* is not analytic at *z = 0*.

1.2.23 Theorem: Real and imaginary parts of an analytic function satisfy Laplace equation.

Solution. Let *f(z) = u + iv* be an analytic function so that C-R equations $u_x = v_y$, $u_y = -v_x$ are satisfied. Differentiating first C-R equation w.r.t. *x* and second w.r.t. *y* and adding, we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y}-\frac{\partial^2 v}{\partial y \partial x}$$

Where continuity of partial derivatives implies that the mixed derivatives are equal i.e. $v_{xy} = v_{yx}$. Hence, we get $u_{xx}+u_{yy}=0$ *i.e.* $\nabla^2 u=0.$

Similarly, differentiating first equation w.r.t y and second w.r.t x and then subtracting, we find

$$\frac{\partial^2 u}{\partial y\partial x}-\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}$$

i.e. $v_{xx}+v_{yy}=0$ *i.e.* $\nabla^2 v=0.$

1.2.24 Remark: C-R equations, in polar from, are

$$u_r=\frac{1}{r}v_\theta,\, v_r=-\frac{1}{r}u_\theta$$

Differentiating first equation w.r.t r and second w.r.t θ, we get

$$v_{\theta r}=u_r+ru_{rr},\, v_{r\theta}=-\frac{1}{r}u_{\theta\theta}$$

Thus, using the continuity of second order partial derivatives, we get

$$u_r+ru_{rr}=-\frac{1}{r}u_{\theta\theta}$$

i.e. $$u_r+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0$$

which is the polar form of Laplace equation.

1.2.25 Definition: A real valued function $u(x, y)$ of real variables x and y is said to be **harmonic** on a domain $D\subset\mathbb{C}$, if for all points (x, y) in D, all second partial derivatives exist, continuous and satisfies Laplace equation. Thus, from the above theorem 1.2.23, we observe that u and v are harmonic functions. In such a case, u and v are called conjugate harmonic functions i.e. u is referred to as the harmonic conjugate of v and vice-versa where $f(z)= u + iv$ is analytic. Harmonic functions play a part in both physics and mathematics.

1.2.26 Definition: Two families of curves $u(x, y) = c_1$ and $v(x, y) = c_2$ are said to form an **orthogonal system** if they intersect at right angle to each other at each of their points of intersection.

1.2.27 Exercise: If $f(z) = u + iv$ is an analytic function in domain D then curves $u(x, y) = c_1$ and $v(x, y) = c_2$ form two orthogonal families.

Solution: Since $f(z) = u + iv$ is analytic, so C-R equations are satisfied. Let m_1 = slope of tangent to the curve $u(x, y)$ and m_2 = slope of tangent to the curve $v(x, y)$. To prove that $u(x, y) = c_1$ and $v(x, y) = c_2$ form two orthogonal families. It is sufficient to show that $m_1 m_2 = -1$.

Differentiate $u(x, y) = c_1$ and $v(x, y) = c_2$, we have

$$\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=0 \quad (1) \qquad \text{and} \quad \frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy=0 \qquad (2)$$

From (1), $m_1=\frac{dy}{dx}=\frac{-\partial u/\partial x}{\partial u/\partial y}=\frac{u_x}{u_y}$

From (2), $m_2 = \frac{dy}{dx} = \frac{-v_x}{v_y} = \frac{u_y}{u_x}$ [using CR equation]

Hence, $m_1 m_2 = -1$.

1.2.28 Exercise: Prove that $u(x,y) = e^{-x}(x\sin y - y\cos y)$ is harmonic and find the conjugate harmonic function $v(x,y)$ such that $f(z) = u + iv$ is analytic.

Solution: Here $u(x,y) = e^{-x}(x\sin y - y\cos y)$

$$\frac{\partial^2 u}{\partial x^2} = e^{-x}(-2\sin y + x\sin y - y\cos y) \tag{1}$$

Similarly, $\frac{\partial^2 u}{\partial y^2} = e^{-x}(2\sin y - x\sin y + y\cos y)$ (2)

From (1) and (2)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \ .$$

Thus, $u(x,y)$ is harmonic.

Now, again $u(x,y) = e^{-x}(x\sin y - y\cos y)$

$$\phi(x,y) = \frac{\partial u}{\partial x} = e^{-x}(\sin y - x\sin y + y\cos y)$$

$$\psi(x,y) = \frac{\partial u}{\partial y} = e^{-x}(y\sin y + x\cos y - \cos y)$$

Now, by Milne Thomson Method,

$$f'(z) = \phi(z,0) - i\psi(z,0)$$

$$= 0 - i\left(ze^{-z} - e^{-z}\right) = -i\left(ze^{-z} - e^{-z}\right)$$

$$\Rightarrow f(z) = -i\int\left(ze^{-z} - e^{-z}\right)dz = i\,ze^{-z} + c$$

Now, if we write $f(z) = u(x,y) + iv(x,y)$, then we get $u(x,y) + iv(x,y) = i\,ze^{-z} + c$

$$\Rightarrow e^{-x}(x\sin y - y\cos y) + iv(x,y) = ize^{-z} + c$$

$$\Rightarrow iv(x,y) = ize^{-z} - e^{-x}(x\sin y - y\cos y) + c$$

$$iv(x,y) = i(x+iy)e^{-(x+iy)} - e^{-x}(x\sin y - y\cos y) + c$$
$$= ie^{-x}\left[xe^{-iy} + iye^{-iy} + i(x\sin y - y\cos y)\right] + c$$
$$= ie^{-x}\left[x(\cos y - i\sin y) + iy(\cos y - i\sin y) + i(x\sin y - y\cos y)\right] + c$$
$$= ie^{-x}\left[x\cos y - ix\sin y + iy\cos y + y\sin y + ix\sin y - iy\cos y\right] + c$$

$$iv(x,y) = ie^{-x}(x\cos y + y\sin y) + c$$
$$\Rightarrow \ v(x,y) = e^{-x}(x\cos y + y\sin y) + \frac{c}{i}$$

Since *v(x, y)* is a real function of real variables *x* and *y* therefore *c* must be of the form $c'i$ where c' is some real constant.

1.2.29 Example: Find the analytic function *f (z) = u + iv* where

$$u+v = \frac{2\sin x}{e^{2y} + e^{-2y} - 2\cos 2x} = \frac{\sin 2x}{\cosh 2y - \cos 2x} \ .$$

Solution: Given *f (z) = u + iv* (1)

$$u+v = \frac{2\sin x}{e^{2y} + e^{-2y} - 2\cos 2x} \qquad (2)$$

$$\Rightarrow if(z) = iu - v \ \text{[By (1)]} \qquad (3)$$

Adding (1) and (3), we have

$$(1+i)f(z) = (u-v) + i(u+v)$$

Let $u-v=U$ and $u+v=V$ then

$$(1+i)f(z) = F(z) = U + iV \ \text{,where } z = u+iv$$

Since *f (z)* is analytic so *F(z)* is also analytic. Now, from Milne Thomson Method, we have

$$F'(z) = U_x + iV_x = V_y + iV_x \ \text{[By C-R equation]}$$
$$= \phi_1(x,y) + i\phi_2(x,y)$$
$$\Rightarrow \quad F(z) = \int\left(\phi_1(z,0) + i\phi_2(z,0)\right)dz + c \ .$$

In the present case $\phi_1(x,y) = \dfrac{\partial V}{\partial y} = \dfrac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$

$$\phi_2(x,y) = \frac{\partial V}{\partial x} = \frac{2\cos 2x(\cosh 2y - \cos 2x) - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2}$$

Hence, $F(z) = (1+i)f(z)$

$$= \int \left[\phi_1(z,0) + i\phi_2(z,0) \right] dz + c$$

$$= -2i \int \frac{dz}{1-\cos 2z} + c$$

$$= -i \int \frac{dz}{\sin^2 z} + c$$

$$= -i \int \operatorname{cosec}^2 z dz + c = i \cot z + c$$

$$\Rightarrow \quad f(z) = \frac{i}{1+i} \cot z + c', \quad c' = \frac{c}{1+i}$$

$$= \frac{i(1-i)}{2} \cot z + c' = \frac{1+i}{2} \cot z + c'.$$

1.2.30 Example: Show that an analytic function with constant modulus is constant.

Solution: Let $f(z) = u + iv$ and $|f(z)|$ = constant = c(say)

We have

$$|u+iv|^2 = c^2$$

$$\Rightarrow \quad u^2 + v^2 = c^2$$

Differentiating partially w.r.t. x, we have

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \qquad (1)$$

Similarly $u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0$ (2)

Using C-R equation $v_y = u_x$ and $v_x = -u_y$ in (2) and (1) respectively. We get

$$u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0 \qquad (3)$$

and $u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial x} = 0$ (4)

Multiply (3) by *u and* (4) *by v* and add

$$\left(u^2 + v^2\right)\frac{\partial u}{\partial x} = 0,$$

But $u^2 + v^2 \neq 0$. Therefore, $\frac{\partial u}{\partial x} = 0$. Similarly $\frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = 0$ and $\frac{\partial v}{\partial y} = 0$.

Thus, all the four first order partial derivatives of u and v are zero. Therefore, function u and v are constant and hence $f(z)$ is constant.

1.3. Power series. An infinite series of the from

$$(i)\sum_{n=0}^{\infty} a_n z^n \qquad or \qquad (ii)\sum_{n=0}^{\infty} a_n\left(z-z_0\right)^n$$

where a_n, z, z_0 are in general complex, is called a power series. Since the series (ii) can be transformed into the series (i) by means of change of origin, it is sufficient to consider only the series of type (i).

1.3.1 Some Tests for convergence of series:

(i) If series $\sum_{n=0}^{\infty} u_n$ is convergent, then $\lim_{n\to\infty} u_n = 0$.

(ii) **Ratio Test:** The series $\sum_{n=0}^{\infty} u_n$ is convergent or divergent according as $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$ or > 1 .

(iii)**Cauchy Root Test:** If $\lim_{n\to\infty}\left|u_n\right|^{1/n} = l$, then the series $\sum_{n=0}^{\infty} u_n$ is convergent or divergent according as $l < 1$ or $l > 1$ and test fails if $l = 1$.

(iv)***P*-Test:** The series $\sum_{n=0}^{\infty}\frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \le 1$.

(v) **Comparison Test:** $\sum_{n=0}^{\infty} u_n$ is absolutely convergent if $|u_n| \le |v_n|$ and $\sum_{n=0}^{\infty} v_n$ is convergent.

1.3.2 Theorem: The power series $\sum_{n=0}^{\infty} a_n z^n$

(i) Converges for every z,

(ii) Converges only for $z = 0$,

(iii)Converges for z in some region, in the complex plane.

Proof: We give an example of each case

(i) The series $\sum_{n=0}^{\infty}\frac{z^n}{\lfloor\underline{n}}$ converges absolutely for all values of z. We have $u_n = \frac{z^n}{\lfloor\underline{n}}$ and $u_{n+1} = \frac{z^{n+1}}{\lfloor\underline{n+1}}$.

$$\lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{z^n}{\lfloor\underline{n}}\cdot\frac{\lfloor\underline{n+1}}{z^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n+1}{z}\right| = \lim_{n\to\infty}\frac{n+1}{|z|} = \infty > 1$$

So, by D-Ratio test the series is absolutely convergent for all values of z.

(ii) The series $\sum_{n=0}^{\infty}\lfloor\underline{n}\, z^n$ converges only for $z = 0$

$$\lim_{n\to\infty}\lfloor\underline{n}\, z^n = \begin{cases}\infty & if\ z \ne 0\\ 0 & if\ z = 0\end{cases}$$

Hence, series is not convergent for $z \ne 0$.

(iii) The geometric series $\sum_{n=0}^{\infty} z^n$ converges for $|z| < 1$ and diverges for $|z| \geq 1$.

1.3.3 Theorem: Power series converges for a particular value z_0 of z, then it converges absolutely for every z for which $|z| < |z_0|$.

Proof: Since $\sum_{n=0}^{\infty} a_n z^n$ converges for a particular value z_0. Therefore, $\sum_{n=0}^{\infty} a_n z_0^n$ converges then its n[th] term $a_n z_0^n$ must tends to zero as $n \to \infty$. So we can find a number $M > 0$ such that

$$|a_n z_0^n| \leq M \text{, for all } n. \text{ Therefore, } |a_n z^n| \leq M \left| \frac{z}{z_0} \right|^n .$$

Since $|z| < |z_0|$. Therefore, series $\sum_{n=0}^{\infty} \left| \frac{z}{z_0} \right|^n$ converges for all values of z for which $|z| < |z_0|$. In otherwords, $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for all z, for which $|z| < |z_0|$.

1.3.4 Circle of Convergence: The circle $|z| = R$ which includes all the values of z for which the power series $\sum_{n=0}^{\infty} a_n z^n$ converges is called the circle of convergence of the series and the radius R of this circle is called radius of convergence of the series.

1.3.5 Theorem (Cauchy-Hadmard Theorem): The series $\sum_{n=0}^{\infty} a_n z^n$, there exist a number R, radius of convergence of power series, converges for $|z| < R$ and divergence for $|z| > R$.

Proof: The series $\sum_{n=0}^{\infty} a_n z^n$ is absolutely convergent if the series $\sum_{n=0}^{\infty} \left| a_n z^n \right|$ is convergent. But the series $\sum_{n=0}^{\infty} |a_n| \left| z^n \right|$ is a series of positive terms. Hence, all the tests for convergence of positive terms can be applied to this series. Thus, if we apply **Cauchy Root Test** we see that power series is absolutely convergent if $\lim_{n\to\infty} |a_n|^{1/n} |z| < 1$. If we put $\lim_{n\to\infty} |a_n|^{1/n} = \frac{1}{R}$. Note that the series is absolutely convergent if $\frac{|z|}{R} < 1$ i.e. $|z| < R$ and the series is divergent if $\frac{|z|}{R} > 1$ i.e. $|z| > R$. Hence proved.

1.3.6 Remark: The radius of convergence of the power series using ratio test or Cauchy's root test, is given by the formula

$$R = \lim_{n\to\infty} |a_n|^{-1/n} = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

or $$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

The number R is unique and R=∞ is allowed, in that case the series converges for arbitrarily large $|z|$. Also this is known as Hadmard's formula for the radius of convergence and hence the above theorem can also be stated as "The power series $\sum_{n=0}^{\infty} a_n z^n$ converges within and diverges outside the circle of radius $R = \lim_{n\to\infty} |a_n|^{-1/n}$ which has its centre at the origin."

1.3.7 Theorem: Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series and let $\sum_{n=1}^{\infty} n a_n z^{n-1}$ be the power series obtained by differentiating the first series term by term then the desired series, has the same radius of convergence as the original series.

Proof: Let R and R' be the radius of convergence of the above given two series respectively. Then,

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n}$$

$$\frac{1}{R'} = \lim_{n\to\infty} |na_n|^{1/n} = \lim_{n\to\infty} n^{1/n} |a_n|^{1/n}$$

If $\lim_{n\to\infty} n^{1/n} = 1$, the result will be over.

To prove this, let $n^{1/n} = 1 + h_n$ so that

$$n = (1+h_n)^n = 1 + nh_n + \frac{n(n-1)}{\underline{|2}} h_n^2 + \ldots + h_n^n .$$

$$\Rightarrow \quad n > \frac{n(n-1)}{2} h_n^2$$

$$\Rightarrow \quad h_n^2 < \frac{2}{n-1} .$$

We obtain $h_n \to 0$ as $n \to \infty$ and therefore, $\lim_{n\to\infty} n^{1/n} = 1$. Hence $R = R'$.

1.3.8 Remark: Our interest in power series is in their behavior as functions. The power series can be used to give examples of analytic functions. A power series $\sum_{n=0}^{\infty} a_n z^n$ with non-zero radius of convergence *R,* converges for $|z| < R$, and so we can define a function *f* by $f(z) = \sum_{n=0}^{\infty} a_n z^n \ (|z| < R)$. The function *f (z)* is called sum **function** of the power series.

1.3.9 Theorem: The sum function *f (z)* of the power series $\sum_{n=0}^{\infty} a_n z^n$ represents an analytic function inside its circle of convergence. Further, every power series possesses derivatives of all order within its circle of convergence and these derivatives are obtained by term by term differentiation of the given power series.

Proof. Let the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ be R and let

$$f(z)=\sum_{n=0}^{\infty} a_n z^n, \phi(z)=\sum_{n=0}^{\infty} n\, a_n z^{n-1}$$

The radius of convergence of the second series is also R. Suppose that z is any point within the circle of convergence so that $|z| < R$. Then there exists a positive number r such that $|z| < r < R$. For convenience, we write $|z| = \rho$, $|h| = \varepsilon$. Then $\rho < R$. Also h may be so chosen that $\rho + \varepsilon < R$.

Since $\sum_{n=0}^{\infty} a_n z^n$ is convergent in $|z|<R$, $\sum_{n=0}^{\infty} a_n r^n$ is bounded for $0 < r < R$ so that $|a_n r^n| < M$ where M is finite positive constant. Thus, we have

$$\left|\frac{f(z+h)-f(z)}{h}-\phi(z)\right| = \left|\sum_{n=0}^{\infty} a_n \left[\frac{(z+h)^n - z^n}{h} - nz^{n-1}\right]\right|$$

$$= \left|\sum_{n=0}^{\infty} a_n z^n \left[\frac{n(n-1)}{2} z^{n-2} h + \ldots + h^{n-1}\right]\right|$$

$$\leq \sum_{n=0}^{\infty} | a_n | \left[\frac{n(n-1)}{2} z^{n-2} h + \ldots + h^{n-1}\right]$$

$$< \sum_{n=0}^{\infty} \frac{M}{r^n}\left[\frac{n(n-1)}{2} \rho^{n-2}\varepsilon + \ldots + \varepsilon^{n-1}\right]$$

$$= \sum_{n=0}^{\infty} \frac{M}{r^n} \in \left[(\rho+\varepsilon)^n - \rho^n - n\rho^{n-1}\varepsilon\right]$$

$$= \frac{M}{\varepsilon} \sum_{n=0}^{\infty} \left[\left(\frac{\rho+\varepsilon}{r}\right)^n - \left(\frac{\rho}{r}\right)^n + n\left(\frac{\rho}{r}\right)^n \frac{\varepsilon}{\rho}\right] \tag{2}$$

Now, $\sum_{n=0}^{\infty} \left(\frac{\rho+\varepsilon}{r}\right)^n = 1 + \frac{\rho+\varepsilon}{r} + \left(\frac{\rho+\varepsilon}{r}\right)^2 + \ldots\ldots = \frac{1}{1-\frac{\rho+\varepsilon}{r}} = \frac{r}{r-\rho-\varepsilon}.$

and

$$\sum_{n=0}^{\infty} \left(\frac{\rho}{r}\right)^n = 1 + \frac{\rho}{r} + \left(\frac{\rho}{r}\right)^2 + \ldots\ldots$$

$$= \frac{1}{1-\frac{\rho}{r}} = \frac{r}{r-\rho}$$

Let us write $S = \sum n \left(\frac{\rho}{r}\right)^n = 1.\frac{\rho}{r} + 2.\left(\frac{\rho}{r}\right)^2 + 3.\left(\frac{\rho}{r}\right)^3 + \ldots\ldots$

Then $$S\frac{\rho}{r}=\left(\frac{\rho}{r}\right)^2+2.\left(\frac{\rho}{r}\right)^3+\ldots\ldots$$

Subtracting, we get

$$S\left(1-\frac{\rho}{r}\right)=\frac{\rho}{r}+\left(\frac{\rho}{r}\right)^2+\ldots..$$

$$=\frac{\rho/r}{1-\rho/r}=\frac{\rho}{r-\rho}$$

or $$S=\frac{\rho r}{(r-\rho)^2}$$

Using the values of these sums, (2) becomes

$$\left|\frac{f(z+h)-f(z)}{h}-\phi(z)\right|<\frac{M}{\varepsilon}\left[\frac{r}{r-\rho-\varepsilon}-\frac{r}{r-\rho}+\frac{\varepsilon r}{(r-\rho)^2}\right]$$

$$=\frac{Mr\varepsilon}{(r-\rho-\varepsilon)(r-\rho)^2}$$

Which tends to zero as $\varepsilon \to 0$.

Hence, $$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}=\phi(z)$$

It follows that $f(z)$ has the derivative $\phi(z)$. Thus, $f(z)$ is differentiable so that $f(z)$ is analytic for $|z| < R$. Again, since the radius of convergence of the derived series is also R, so $\phi(z)$ is also analytic in $|z| < R$. Successively differentiating and applying the theorem, we see that the sum function $f(z)$ of a power series possesses derivatives of all orders within its circle of convergence and all these derivatives are obtained by term by term differentiation of the series.

In other words, a power series represents an analytic function inside its circle of convergence.

1.3.10 Example: Find the radius of convergence of the following power series:

(i) $\displaystyle\sum_{n=0}^{\infty}\frac{z^n}{n^n}$ (ii) $\displaystyle\sum_{n=0}^{\infty}\frac{2^{-n}z^n}{1+in^2}$ (iii) $\displaystyle\sum_{n=0}^{\infty}\frac{(\underline{|n})^2 z^n}{\underline{|2n}}$

Solution: (i) *Compare* $\displaystyle\sum_{n=0}^{\infty}\frac{z^n}{n^n}$ *with* $\displaystyle\sum_{n=0}^{\infty}a_n z^n$, $a_n=\dfrac{1}{n^n}$

$$\frac{1}{R}=\lim_{n\to\infty}|a_n|^{1/n}=\lim_{n\to\infty}\left|\frac{1}{n^n}\right|^{1/n}=\lim_{n\to\infty}\frac{1}{n}=\frac{1}{\infty}=0.$$

$$\Rightarrow R=\infty$$

(ii) Here, $a_n=\dfrac{2^{-n}}{1+in^2}$

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\frac{2^{-n}}{1+in^2}\right|^{1/n} = \lim_{n\to\infty} \left(\frac{2^{-n}}{\sqrt{1+n^4}}\right)^{1/n}$$

$$= \lim_{n\to\infty} \frac{1}{2\left(1+n^4\right)^{1/2n}} = \lim_{n\to\infty} \frac{1}{2\left(n^4\left(1+\frac{1}{n^4}\right)\right)^{1/2n}}$$

$$= \lim_{n\to\infty} \frac{1}{2n^{2/n}}\left(1+\frac{1}{n^4}\right)^{-1/2n} = \frac{1}{2}$$

$\Rightarrow \quad R = 2$

(iii) Here, $a_n = \dfrac{(\lfloor n)^2}{\lfloor 2n}$. Therefore, $\dfrac{1}{R} = \lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \dfrac{(\lfloor n+1)^2 \lfloor 2n}{\lfloor 2n+2 (\lfloor n)^2}$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{(n+1)}{2(2n+1)} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{4\left(1+\frac{1}{2n}\right)} = \frac{1}{4}$$

$\therefore R = 4$

1.3.11 Example: Find R for the following series:

(a) $\displaystyle\sum_{n=0}^{\infty} (3+4i)^n z^n$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (z-2i)^n}{n}$

(c) $\displaystyle\sum_{n=1}^{\infty} (\log n)^n z^n$ (d) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{\sqrt{2n}+i}{1+2i\,n}\right) z^n$

Solution: (a) Here $a_n = (3+4i)^n$

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|(3+i4)^n\right|^{1/n} = \lim_{n\to\infty} |3+i4| = \sqrt{9+16} = 5.$$

$\Rightarrow R = 5$

(b) Series is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (z-2i)^n}{n}$ comparing with $\displaystyle\sum_{n=1}^{\infty} a_n (z-a)^n$.

Here $a = 2i$, which is the centre of the circle of convergence

$$\therefore a_n = \frac{(-1)^n}{n}, a_{n+1} = \frac{(-1)^{n+1}}{n+1}$$

$$\therefore \frac{a_{n+1}}{a_n} = -\left(\frac{n}{n+1}\right)$$

$$\frac{1}{R}=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{n}{n+1}=\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}=1$$

Hence, $R=1$.

(c) Here, $a_n=\left(\log n\right)^n$

$$\frac{1}{R}=\lim_{n\to\infty}|a_n|^{1/n}=\lim_{n\to\infty}\left[\left(\log n\right)^n\right]^{1/n}=\lim_{n\to\infty}\log n=\infty.$$

$\Rightarrow R=0$

(d) Here, $a_n=\dfrac{\sqrt{2n}+i}{1+2i\,n}$

$$\left|a_n\right|=\left|\frac{\sqrt{2n}+i}{1+2i\,n}\right|=\sqrt{\frac{2n^2+1}{4n^2+1}}=\frac{1}{\sqrt{2}}\left[\frac{1+\frac{1}{2n^2}}{1+\frac{1}{4n^2}}\right]^{1/2}$$

$$\therefore\frac{1}{R}=\lim_{n\to\infty}|a_n|^{1/n}=\lim_{n\to\infty}\frac{1}{2^{1/2n}}\left[\frac{1+\frac{1}{2n^2}}{1+\frac{1}{4n^2}}\right]^{1/2n}=\frac{1}{2^0}.1=1.$$

$\Rightarrow R=1$

1.3.12 Example: Find that the series

$$1+\frac{a.b}{1.c}z+\frac{a.(a+1)b(b+1)}{1.2.c(c+1)}z^2+\ldots$$

has unit radius of convergence.

Proof: Here, $a_n=\dfrac{a(a+1)\ldots(a+n-1)b(b+1)\ldots(b+n-1)}{1.2\ldots n.c(c+1)\ldots(c+n-1)}$

$$a_{n+1}=\frac{a(a+1)\ldots(a+n-1)(a+n)b(b+1)\ldots(b+n-1)(b+n)}{1.2\ldots n.(n+1).c(c+1)\ldots(c+n-1)(c+n)}$$

$$\frac{a_{n+1}}{a_n}=\frac{\left(1+\frac{a}{n}\right)\left(1+\frac{b}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{c}{n}\right)}$$

$$\therefore\frac{1}{R}=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1\Rightarrow R=1.$$

1.3.13 Example: Find the radius of convergence of the series

$$\frac{z}{2}+\frac{1.3}{2.5}z^2+\frac{1.3.5}{2.5.8}z^3+\ldots$$

Solution: Here, $a_n=\dfrac{1.3.5\ldots(2n-1)}{2.5.8\ldots(3n-1)}$

$$a_{n+1}=\frac{1.3.5\ldots(2n-1)(2n+1)}{2.5.8\ldots(3n-1)(3n+2)}$$

$$\frac{a_{n+1}}{a_n}=\frac{2+\frac{1}{n}}{3+\frac{2}{n}} \qquad \therefore \frac{1}{R}=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\frac{2}{3}$$

$$\Rightarrow R=\frac{3}{2}.$$

1.3.14 Example: Find the domain of convergence of the series $\sum_{n=0}^{\infty} n^2\left(\dfrac{z^2+1}{1+i}\right)^n$.

Solution: Consider the transformation $z^2=\eta$ then, the series becomes

$$\sum_{n=0}^{\infty} n^2\left(\frac{\eta+1}{1+i}\right)^n=\sum_{n=0}^{\infty}U_n(\eta) \text{ (say)}$$

$$\frac{U_{n+1}}{U_n}=\frac{(n+1)^2}{n^2}\left(\frac{\eta+1}{1+i}\right)^{n+1}\left(\frac{1+i}{\eta+1}\right)^n=\left(\frac{n+1}{n}\right)^2\left(\frac{1+\eta}{1+i}\right)$$

$$\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right|=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2\left|\frac{1+\eta}{1+i}\right|=\frac{|\eta+1|}{\sqrt{2}}$$

Now, the given series is convergent if

$$\lim_{n\to\infty}\left|\frac{U_{n+1}}{\mathrm{U}_n}\right| < 1 \textit{ i.e.} \frac{|\eta+1|}{\sqrt{2}} < 1$$

i.e. $|\eta+1|<\sqrt{2}$

i.e. $|z^2+1|<\sqrt{2}$,which gives domain of convergence.

1.3.15 Example: Examine the behavior of the power series

$$\sum_{n=2}^{\infty}\frac{z^n}{n(\log n)^2} \text{ on the circle of convergence.}$$

Solution: Here $a_n = \dfrac{1}{n(\log n)^2}$ and $a_{n+1} = \dfrac{1}{(n+1)(\log(n+1))^2}$

$$\therefore \frac{a_{n+1}}{a_n} = \frac{n(\log n)^2}{(n+1)[\log(n+1)]^2} = \frac{1}{\left(1+\frac{1}{n}\right)\left[\frac{\log n\left(1+\frac{1}{n}\right)}{\log n}\right]^2} = \frac{1}{\left(1+\frac{1}{n}\right)\left[\frac{\log n+\log\left(1+\frac{1}{n}\right)}{\log n}\right]^2}$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)}\left[\frac{1}{1+\frac{\log\left(1+\frac{1}{n}\right)}{\log n}}\right]^2 = \frac{1}{\left(1+\frac{1}{n}\right)}\left[\frac{1}{1+\frac{1}{n\log n}-\frac{1}{2n^2\log n}+\ldots}\right]^2$$

$$\therefore \frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$$

$$\Rightarrow R = 1$$

The radius of convergence of the series is one and centre is at $z = 0$.

For every point on the circle of convergence, we have

$$\left|\frac{z^n}{n(\log n)^2}\right| = \frac{1}{n(\log n)^2} \text{, on the circle } |z| = 1$$

Also $\sum \dfrac{1}{n(\log n)^2}$ is convergent by **Cauchy condensation test**. Hence, the given series $\sum \dfrac{z^n}{n(\log n)^2}$ is absolutely convergent for all z on the circle of convergence.

1.3.16 Example: Find the domain of convergence of the following series

(i) $\displaystyle\sum_{i=1}^{\infty} \frac{1.3.5\ldots(2n-1)}{\lfloor n}\left(\frac{1-z}{z}\right)^n$

(ii) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{iz-1}{2+i}\right)^n$

Solution: (i) Putting $\dfrac{1}{z} = \eta$, the series becomes $\displaystyle\sum_{i=1}^{\infty}\frac{1.3.5\ldots(2n-1)}{\lfloor n}(\eta-1)^n$

Here $a_n = \dfrac{1.3.5\ldots(2n-1)}{\lfloor n}$

$$\therefore \frac{1}{R} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left| \frac{2\left(1+\frac{1}{2n}\right)}{1+\frac{1}{n}} \right|^2 = 2$$

$$\Rightarrow R = \frac{1}{2}$$

Now the domain of convergence is given by

$$|\eta - 1| < \frac{1}{2} \text{ i.e. } \left|\frac{1}{z} - 1\right| < \frac{1}{2}$$

or $\quad 2|1-z| < |z|$

$\Rightarrow \quad 4|1-z|^2 < |z|^2$

$$4\left[(1-z)(1-\bar{z})\right] < z.\bar{z}$$

$\Rightarrow \quad 4\left[1 + z.\bar{z} - z - \bar{z}\right] < z.\bar{z}$

$\Rightarrow \quad 4 + 3.z.\bar{z} - 4(z+\bar{z}) < 0$

$\Rightarrow \quad z.\bar{z} - \frac{4}{3}(z+\bar{z}) + \frac{4}{3} < 0$

$\Rightarrow \quad \left(z-\frac{4}{3}\right)\left(\bar{z}-\frac{4}{3}\right) < \left(\frac{4}{3}\right)^2 - \frac{4}{3} = \frac{4}{9}$

$\Rightarrow \quad \left|z-\frac{4}{3}\right|^2 < \left(\frac{2}{3}\right)^2 \Rightarrow \quad \left|z-\frac{4}{3}\right| < \left(\frac{2}{3}\right)$

This represents a circle with centre at 4/3 and radius is 2/3. Thus, the series is convergent inside the circle.

(ii) Take $\sum_{n=0}^{\infty}\left(\frac{iz-1}{2+i}\right)^n = \sum u_n(z)$ (say)

$$u_n = \left(\frac{iz-1}{2+i}\right)^n \Rightarrow \frac{u_{n+1}}{u_n} = \frac{iz-1}{2+i}$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{iz-1}{2+i}\right|$$

$$= \lim_{n\to\infty}\frac{|i|.|z+i|}{\sqrt{5}} = \frac{|z+i|}{\sqrt{5}}.$$

Thus, the series is convergent if $\frac{|z+i|}{\sqrt{5}}<1$ i.e. $|z+i|<\sqrt{5}$ .Therefore, the power series is convergent for all values of z which lie inside the circle of radius $\sqrt{5}$ and centre at $z=-i$.

1.4. Multivalued Function and its Branches: The familiar fact that $\sin\,\theta$ and $\cos\,\theta$ are periodic functions with period 2π, is responsible for the non-uniqueness of θ in the representation $z = |z|e^{i\theta}$ i.e. $z = re^{i\theta}$. Here, we shall discuss non- uniqueness problems with reference to the function $\arg z$, $\log z$ and z^a. We know that a function $w = f(z)$ is multivalued for given z, we may find more than one value of w. Thus, a function $f(z)$ is said to be single-valued if it satisfies

$$f(z) = f(z(r, \theta)) = f(z(r, \theta + 2\pi))$$

otherwise it is classified as multivalued function.

For analytic properties of a multivalued function, we consider domains in which these functions are single valued. This leads to the concept of branches of such functions. Before discussing branches of a many valued function, we give a brief account of the three functions $\arg z$, $\log z$ and z^a.

1.4.1 Argument Function: For each $z \in \mathbb{C}$, $z \neq 0$, we define the argument of z to be

$$\arg z = [\arg z] = \{\theta \in \mathbb{R} : z = |z|e^{i\theta}\}$$

the square bracket notation emphasizes that $\arg z$ is a set of numbers and not a single number i.e. $[\arg z]$ is multivalued. In fact, it is an infinite set of the form $\{\theta+2n\pi : n \in I\}$, where θ is any fixed number such that $e^{i\theta} = \frac{z}{|z|}$. For example $\arg i = \{(4n + 1)\pi/2 : n \in I\}$

Also, $\arg(1/z) = \{-\theta : \theta \in \arg z\}$

Thus, for $z_1, z_2 \neq 0$, we have

$$\arg(z_1z_2) = \{\theta_1 + \theta_2 : \theta_1 \in \arg z_1, \theta_2 \in \arg z_2\}$$
$$= \arg z_1 + \arg z_2$$

and $$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$$

For principal value determination, we can use $\operatorname{Arg} z = \theta$, where $z = |z|\, e^{i\theta}$, $-\pi < \theta \leq \pi$ (or $0 \leq \theta < 2\pi$). When z performs a complete anticlockwise circuit round the unit circle, θ increases by 2π and a jump discontinuity in $\operatorname{Arg} z$ is inevitable. Thus, we cannot impose a restriction which determines θ uniquely and therefore for general purpose, we use more complicated notation $\arg z$ or $[\arg z]$ which allows z to move freely about the origin with θ varying continuously. We observe that

$$\arg z = [\arg z] = \operatorname{Arg} z + 2n\pi,\ n \in I.$$

1.4.2 Logarithmic Function. We observe that the exponential function e^z is a periodic function with a purely imaginary period of 2πi, since

$$e^{z+2\pi i} = e^z.e^{2\pi i} = e^z,\ e^{2\pi i} = 1.$$

i.e. $\exp(z+2\pi i) = \exp z$ for all z.

If w is any given non-zero point in the w-plane then there is an infinite number of points in the

z-plane such that the equation

$$w = e^z \tag{1}$$

is satisfied. For this, we note that when z and w are written as $z = x + iy$ and $w = \rho e^{i\phi}$ $(-\pi < \phi \le \pi)$, equation (1) can be put as

$$e^z = e^{x+iy} = e^x e^{iy} = \rho e^{i\phi} \tag{2}$$

From here, $e^x = \rho$ and $y = \phi + 2n\pi$, $n \in I$.

Since the equation $e^x = \rho$ is the same as $x = \log_e \rho = \log \rho$ (base e understood), it follows that when $w = \rho\, e^{i\phi}$ $(-\pi < \phi \le \pi)$, equation (1) is satisfied if and only if z has one of the values

$$z = \log \rho + i(\phi + 2n\pi),\ n \in I \tag{3}$$

Thus, if we write

$$\log w = \log \rho + i(\phi + 2n\pi),\ n \in I \tag{4}$$

we see that $\exp(\log w) = w$, this motivates the following definition of the (multivalued) logarithmic function of a complex variable.

The logarithmic function is defined at non-zero points $z = re^{i\theta}$ $(-\pi < \theta \le \pi)$ in the z-plane as

$$\log z = \log r + i\,(\theta + 2n\pi),\ n \in I \tag{5}$$

The principal value of log z is the value obtained from (5) when n = 0 and is denoted by Log z. Thus

$$\mathrm{Log}\, z = \log r + i\,\theta \text{ i.e. } \mathrm{Log}\, z = \log |z| + i\, \mathrm{Arg} z \tag{6}$$

Also, from (5) & (6), we note that

$$\log z = \mathrm{Log}\, z + 2n\pi i,\ n \in I \tag{7}$$

The function Log z is evidently well defined and single-valued when $z \neq 0$.

Equation (5) can also be put as

$$\log z = \{\log |z| + i\theta : \theta \in \arg z\}$$

or $$[\log z] = \{\log |z| + i\theta : \theta \in [\arg z]\} \tag{8}$$

or $$\log z = \log |z| + i\theta = \log |z| + i \arg z \tag{9}$$

where $\theta = \theta + 2n\pi$, $\theta = \mathrm{Arg} z$.

From (8), we find that

$$\log 1 = \{2n\pi i,\ n \in I\},\ \log(-1) = \{(2n+1)\,\pi i,\ n \in I\}$$

In particular, $\mathrm{Log}\ 1 = 0$, $\mathrm{Log}\,(-1) = \pi i$.

Similarly, $\log i = \{(2n+1/2)\,\pi i,\ n \in I\}$, $\log(-i) = \{2n-1/2)\,\pi i,\ n \in I\}$.

In particular, $\mathrm{Log}\, i = \pi i/2$, $\mathrm{Log}\,(-i) = -\pi i/2$.

Thus, we conclude that complex logarithm is not a bonafide function, but a multifunction. We have assigned to each $z \neq 0$ infinitely many values of the logarithm.

1.4.3 Complex Exponents: When $z \neq 0$ and the exponent a is any complex number, the function z^a is defined by the equation.

$$w = z^a = e^{\log z^a} = e^{a \log z} = \exp\left(a \log z\right) \tag{1}$$

Where log z denotes the multivalued logarithmic function. Equation (1) can also be expressed as

$$w = z^a = \{e^{a(\log|z|+i\theta)} : \theta \in \arg z\}$$

or $$\left[z^a\right] = \{e^{a(\log|z|+i\theta)} : \theta \in [\arg z]\}$$

Thus, many valued nature of the function *log z* will generally result in the many-valuedness of z^a. Only when 'a' is an integer, z^a does not produce multiple values. In this case z^a contains a single point z^n. When $a = \frac{1}{n}$, *(n = 2, 3, ...),* then

$$w = z^{1/n} = \left(re^{i\theta}\right)^{1/n} = r^{1/n} e^{i(\theta + 2m\pi)/n}, m \in I$$

We note that in particular, the complex nth roots of ± 1 are obtained as

$$w^n = 1 \Rightarrow w = e^{2m\pi i/n}, w^n = -1 \Rightarrow w = e^{(2m+1)\pi i/n}, m = 0, 1, \ldots n-1$$

for example, $i^{-2i} = \exp(-2i \log i) = \exp[-2i\,(4n+1)\pi i/2]$

$$= \exp\,[(4n+1)\pi],\ n \in I$$

In should be observed that the formula

$$x^a x^b = x^{a+b},\ x,\ a,\ b, \in R$$

can be shown to have a complex analogue (in which values of the multi-functions involved have to be appropriately selected) but the formula $x_1^a\ x_2^a = (x_1\ x_2)^a,\ x_1,\ x_2,\ a \in R$ has no universally complex generalization.

1.4.4 Branches, Branch Points and Branch Cuts: We recall that a multifunction w defined on a set $S \subseteq \mathbb{C}$ is an assignment to each $z \in S$ of a set $[w(z)]$ of complex numbers. Our main aim is that given a multifunction w defined on S, can we select, for each $z \in S$, a point $f(z)$ in $[w(z)]$ so that $f(z)$ is analytic in an open subset G of S, where G is to be chosen as large as possible? If we are to do this, then $f(z)$ must vary continuously with z in G, since an analytic function is necessarily continuous.

Suppose w is defined in some punctured disc D having centre a and radius R i.e. $0 < |z - a| < R$ and that $f(z) \in [w(z)]$ is chosen so that $f(z)$ is at least continuous on the circle γ with centre a and radius r $(0 < r < R)$. As z traces out the circle γ starting from, say z_0, $f(z)$ varies continuously, but must be restored to its original value $f(z_0)$ when z completes its circuit, since $f(z)$ is, by hypothesis, single valued. Notice also that if $z - a = R\ e^{i\theta(z)}$, where $\theta(z)$ is chosen to vary continuously with z, then $\theta(z)$ increases by 2π as z performs its circuit, so that $\theta(z)$ is not restored to its original value. The same phenomenon does not occur if z moves round a circle in the punctured disc D not containing a, in this case θ (z) does return to its original value. More generally, our discussion suggests that if we are to extract an analytic function from a multi- function w, we shall meet to restrict to a set in which it is impossible to encircle, one at a time, points a such that the definition of $[w(z)]$ involves the argument of $(z - a)$. In some cases, encircling several of these "bad" points simultaneously may be allowable.

A branch of a multiple-valued function $f(z)$ defined on $S \subseteq \mathbb{C}$ is any single-valued function $F(z)$ which is analytic in some domain $D \subset S$ at each point of which the value $F(z)$ is one of the values of $f(z)$. The requirement of analyticity, of course, prevents $F(z)$ from taking on a random selection of the values of $f(z)$.

A branch cut is a portion of a line or curve that is introduced in order to define a branch *F(z)* of a multiple-valued function *f(z)*.

A multivalued function *f(z)* defined on $S \subseteq \mathbb{C}$ is said to have a branch point at z_0 when z describes an arbitrary small circle about z_0, then for every branch *F(z)* of *f(z)*, *F(z)* does not return to its original value. Points on the branch cut for *F(z)* are singular points of *F(z)* and any point that is common to all branch cuts of *f(z)* is called a branch point. For example, let us consider the logarithmic function

$$\log z = \log r + i\theta = \log |z| + i\arg z \tag{1}$$

If we let α denote any real number and restrict the values of θ in (1) to the interval $\alpha < \theta < \alpha + 2\pi$, then the function

$$\log z = \log r + i\theta \ (r > 0,\ \alpha < \theta < \alpha + 2\pi) \tag{2}$$

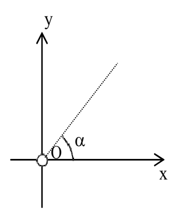
with component functions

$$u(r, \theta) = \log r \text{ and } v(r, \theta) = \theta \tag{3}$$

is single-valued, continuous and analytic function. Thus for each fixed α, the function (2) is a branch of the function (1). We note that if the function (2) were to be defined on the ray θ = α, it would not be continuous there. For, if z is any point on that ray, there are points arbitrarily close to z at which the values of v are near to α and also points such that the values of v are near to $\alpha + 2\pi$. The origin and the ray $\theta = \alpha$ make up the branch cut for the branch (2) of the logarithmic function. The function

$$\operatorname{Log} z = \log r + i\,\theta \ (r > 0,\ -\pi < \theta < \pi) \tag{4}$$

is called the principal branch of the logarithmic function in which the branch cut consists of the origin and the ray $\theta = \pi$. The origin is evidently a branch point of the logarithmic function.



For analyticity of (2), we observe that the first order partial derivatives of u and v are continuous and satisfy the polar form

$$u_r = \frac{1}{r} v_\theta, \qquad v_r = -\frac{1}{r} u_\theta$$

of the C-R equations. Further

$$\frac{d}{dz}(\log z) = e^{-i\theta}(u_r + iv_r)$$
$$= e^{-i\theta}\left(\frac{1}{r} + i0\right) = \frac{1}{re^{i\theta}}$$

Thus, $$\frac{d}{dz}(\log z) = \frac{1}{z}(|z| = r > 0, \alpha < \arg z < \alpha + 2\pi)$$

In particular

$$\frac{d}{dz}(\log z) = \frac{1}{z}(|z| > 0, -\pi < A\operatorname{rg} z < \pi).$$

Further, since *log* $\frac{1}{z}$=-*log z,* ∞ is also a branch point of *log z.* Thus a cut along any half-line from *0* to ∞ will serve as a branch cut.

Now, let us consider the function $w = z^a$ in which a is an arbitrary complex number. We can write

$$w = z^a = e^{\log z^a} = e^{a \log z} \tag{5}$$

where many-valued nature of *log z* results is many-valuedness of z^a. If *Log z* denotes a definite branch, say the principal value of *log z,* then the various values of z^a will be of the form

$$z^a = e^{a(Log\, z + 2n\pi i)} = e^{a\, Log\, z} e^{2n\pi ia} \tag{6}$$

where $\log z = Log\, z + 2n\pi i,\ n \in I.$

The function in (6) has infinitely many different values. But the number of different values of z^a will be finite in the cases in which only a finite number of the values $e^{2\pi ian}$, $n \in I$, are different from one another. In such a case, there must exist two integers *m* and m' $(m' = m)$ such that

$$e^{2\pi iam} = e^{2\pi iam'} \; or \; e^{2\pi ia(m-m')} = 1.$$

Since $e^z = 1$ only if $z = 2\pi in$, thus we get a $(m - m') = n$ and therefore it follows that a is a rational number. Thus z^a has a finite set of values iff *a* is a rational number. If *a* is not rational, z^a has infinity of values.

We have observed that if $z = re^{i\theta}$ and α is any real number, then the branch

$$\log z = \log r + i\theta \; (r > 0 ,\ \alpha < \theta < \alpha + 2\pi) \tag{7}$$

of the logarithmic function is single-valued and analytic in the indicated domain. When this branch is used, it follows that the function (5) is single valued and analytic in the said domain.

The derivative of such a branch is obtained as

$$\frac{d}{dz}(z^a) = \frac{d}{dz}[\exp(a \log z)] = \exp(a \log z)\frac{a}{z}$$
$$= a\frac{\exp(a \log z)}{\exp(\log z)} = a \exp[(a-1)\log z]$$
$$= az^{a-1}$$

As a particular case, we consider the multivalued function $f(z) = z^{1/2}$ and we define

$$z^{1/2} = \sqrt{r} e^{i\theta/2},\ r > 0, \alpha < \theta < \alpha + 2\pi \tag{8}$$

Where the component functions

$$u(r,\theta) = \sqrt{r}\cos\theta/2, v(r,\theta) = \sqrt{r}\sin\theta/2 \tag{9}$$

are single valued and continuous in the indicated domain. The function is not continuous on the line $\theta = \alpha$ as there are points arbitrarily close to z at which the values of v (r, θ) are nearer to $\sqrt{r}\sin\alpha/2$ and also points such that the values of *v(r, θ)* are nearer to $-\sqrt{r}\sin\alpha/2$. The functions (8) is differential as C-R equations in polar from are satisfied by the functions in (9) and

$$\frac{d}{dz}\left(z^{1/2}\right)=e^{-i\theta}\left(u_r+iv_r\right)=e^{i\theta}\left(\frac{1}{2\sqrt{r}}\cos\theta/2+i\frac{1}{2\sqrt{r}}\sin\theta/2\right)$$

$$=\frac{1}{2\sqrt{r}}e^{-i\theta/2}=\frac{1}{2z^{1/2}}$$

Thus (8) is a branch of the function $f(z) = z^{1/2}$ where the origin and the line $\theta = \alpha$ form branch cut. When moving from any point $z = re^{i\theta}$ about the origin, one complete circuit to reach again, at *z,* we have changed *arg z* by 2π. For original function $z = re^{i\theta}$, we have

$$w=\sqrt{r}\ e^{i\theta/2} \text{ and after one complete circle, } w=\sqrt{r}e^{i(\theta+2\pi)/2}=-\sqrt{r}e^{i\theta/2}.$$

Thus, *w* has not returned to its original value and hence change in branch has occurred. Since a complete circuit about $z = 0$ changed the branch of the function, $z = 0$ is a branch point for the function $z^{1/2}$.

SECTION-II

2.1 Complex Integration:

Let [a, b] be a closed interval, where a, b are real numbers. Divide [a, b] into subintervals

$$[a = t_0, t_1], [t_1, t_2], \ldots, [t_{n-1}, t_n = b] \tag{1}$$

by inserting n−1 points $t_1, t_2, \ldots, t_{n-1}$ satisfying the inequalities

$$a = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = b$$

Then the set P = $\{t_0, t_1, \ldots, t_n\}$ is called the **partition** of the interval [a, b] and the greatest of the numbers $t_1 - t_0$, $t_2 - t_1, \ldots, t_n - t_{n-1}$ is called the norm of the partition P. Thus, the norm of the partition P is the maximum length of the subintervals in (1).

2.1.1 Arcs and Curves in the Complex Plane:

An arc (path) L in a region $G \subset \mathbb{C}$ is a continuous function $z(t):[a, b] \to G$ for $t \varepsilon [a, b]$ in $\mathbb{R}$. The arc L, given by z(t) = x(t) + iy(t), t ε [a, b], where x(t) and y(t) are continuous functions of t, is therefore a set of all image points of a closed interval under a continuous mapping. The arc L is said to be differentiable if z′(t) exists for all t in [a, b]. In addition to the existence of z′(t), if $z'(t) : [a, b] \to \mathbb{C}$ is continuous, then z(t) is a **smooth arc**. In such case, we may say that L is regular and smooth. **Thus a regular arc is characterized by the property that x′(t) and y′(t) exist and are continuous over the whole range of values of t.**

We say that an arc is simple or **Jordan arc** if $z(t_1) = z(t_2)$ only when $t_1 = t_2$ i.e. the arc does not intersect itself. If the points corresponding to the values a and b coincide, the arc is said to be a **closed arc** (closed curve). An arc is said to be piecewise continuous in [a,b] if it is continuous in every subinterval of [a, b].

2.1.2 Rectifiable Arcs: Let z = x(t) + iy(t) be the equation of the Jordan arc L, the range for the parameter t being $t_0 \leq t \leq T$. Let $z_0, z_1, \ldots, z_n$ be the points of this arc corresponding to the values $t_0, t_1, \ldots, t_n$ of t, where $t_0 < t_1 < t_2 < \ldots < t_n = T$. Evidently, the length of the polygonal arc obtained by joining successively z_0 and z_1, z_1 and z_2 etc. by straight line segments is given by

$$\begin{aligned}\Sigma_n &= \sum_{r=1}^{n} |z_r - z_{r-1}| \\ &= \sum_{r=1}^{n} |(x_r + iy_r) - (x_{r-1} + iy_{r-1})| \\ &= \sum_{r=1}^{n} |(x_r - x_{r-1}) + i(y_r - y_{r-1})| \\ &= \sum_{r=1}^{n} [(x_r - x_{r-1})^2 + (y_r - y_{r-1})^2]^{1/2}\end{aligned}$$

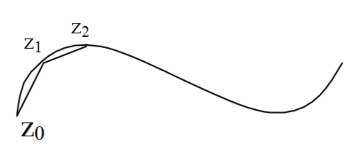


Figure 1

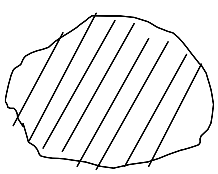
If this sum Σ_n tends to a unique limit $l<\infty$, as $n\to\infty$ and the maximum of the differences $t_r - t_{r-1}$(r =1, 2,…, n) tends to zero, we say that the arc L defined by $z = x(t) + iy(t)$ is rectifiable and that its length is l. In this connection, we have the following result.

"A regular arc $z = x(t) + iy(t)$, $t_0 \le t \le T$ is rectifiable and its length is $\int_{t_0}^{T} [(x'(t))^2 + (y'(t))^2]^{1/2} dt$."

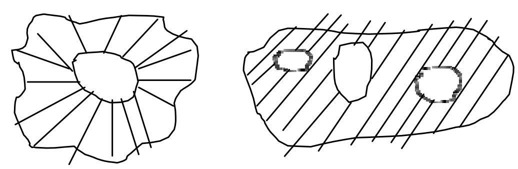
2.1.3 Contours: Let PQ and QR to be two rectifiable arcs with only Q as common point, then the arc PR is evidently rectifiable and its length is the sum of lengths of PQ and QR. Thus, it follows that Jordan arc which consists of a finite number of regular arcs is rectifiable, its length being the sum of lengths of regular arcs of which it is composed. Such an arc is called contour. **Thus a contour C is continuous chain of finite number of regular arcs.** i.e. a contour is a piecewise smooth arc.

By a closed contour, we shall mean a simple closed Jordan arc consisting of a finite number of regular arcs i.e. contour is closed and does not intersect. Clearly, every closed contour is rectifiable. Circle rectangle, triangle, ellipse etc. are examples of closed contour.

2.1.4 Simply Connected Region: Aregion D is said to be simply connected if every simple closed contour within it encloses only points of D. In such a region every closed curve can be shrunk (contracted) to a point without passing out of the region (Figure 2). If the region is not simply connected, then it is called multiply connected (Figure 3)



Simply connected region (**Figure2**)



Multiply connected regions (**Figure3**)

2.1.5 Riemann's Definition of Complex Integration: First, we define the integral as the limit of a sum and later on, deduce it as the operation inverse to that of differentiation.

Let us consider a function $f(z)$ of the complex variable z. We assume that $f(z)$ has a definite value at each point of a rectifiable arc L having equation

$$z(t) = x(t) + iy(t),\ t_0 \le t \le T.$$

We divide this arc L into n smaller arcs by points $z_0, z_1, z_2, \ldots, z_{n-1}, z_n$ (= Z, say) which correspond to the values $t_0 < t_1 < t_2, \ldots, < t_{n-1} < t_n$ (= T) of the parameter t and then form the sum

$$\Sigma = \sum_{r=1}^{n} f(\xi_r)\,(z_r - z_{r-1}),$$

where ξ_r is a point of L between z_{r-1} and z_r. If this sum Σ tends to a unique limit I as $n\to\infty$ and the maximum of the differences $t_r - t_{r-1}$ tends to zero, we say that $f(z)$ is integrable from z_0 to Z along the arc L, and we write

$$I = \int_L f(z)\, dz\,.$$

The direction of integration is from z_0 to Z, since the points on $x(t) + iy(t)$ describe the arc L in this sense when tincreases.

2.1.6 Remarks:

(i) Some of the most obvious properties of real integrals extend at once to complex integrals, forexample,

$$\int_L [f(z)+g(z)]dz = \int_L f(z)dz + \int_L g(z)\ dz,$$

$$\int_L Kf(z)\mathrm{dz} = K\int_L f(z)dz,\ \text{K being constant},$$

$$\int_{L'} f(z)\mathrm{dz} = -\int_L f(z)dz,$$

where L′ denotes the arc L described in opposite direction.

(ii) In the above definition of the complex integral, although z_0, Z play much the same parts as the lower and upper limits in the definite integral of a function of a real variable, we do notwrite

$$I = \int_{z_0}^{z} f(z)dz$$

This is dictated essentially by the fact that the value of I depends, in general, not only on the initial and final points of the arc L but also on its actual form.

In special circumstances, the integral may be independent of path from z_0 to z as shown in the following example.

2.1.7 Example: Using the definition of an integral as the limit of a sum, evaluate the integrals

$$(i)\ \int_L d\,z \qquad (ii)\ \int_L |d\,z| \qquad (iii)\ \int_L z\ d\,z$$

where L is a rectifiable arc joining the points z = α and z = β.

Solution: We first observe that the integrals exist since the integrand is continuous on L in each case.

(i) By definition, we have

$$\int_L dz = \lim_{n\to\infty} \sum_{r=1}^{n} (z_r - z_{r-1})$$

$$= \lim_{n\to\infty} [z_1 - z_0 + z_2 - z_1 + \ldots + z_n - z_{n-1}]$$

$$= \lim_{n\to\infty} (z_n - z_0) = \beta - \alpha.$$

(ii)
$$\int_L |dz| = \lim_{n\to\infty} \sum_{r=1}^{n} |(z_r - z_{r-1})|$$

$$= \lim_{n\to\infty} \left[|z_1 - z_0| + |z_2 - z_1| + \ldots + |z_n - z_{n-1}|\right]$$

= Arc length of L= ℓ (say)

(iii) $\quad Let\, I=\int\limits_L z\,dz=\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\left(z_r-z_{r-1}\right)\xi_r \qquad (1)$

where ξ_r is any point on the sub arc joining z_{r-1} and z_r.

Since ξ_r is arbitrary, we set $\xi_r = z_r$ and $\xi_{r-1} = z_{r-1}$ successively in (1) to find

$$I=\lim_{n\to\infty}\sum_{r=1}^{n} z_r\left(z_r-z_{r-1}\right)$$

$$I=\lim_{n\to\infty}\sum_{r=1}^{n} z_{r-1}\left(z_r-z_{r-1}\right).$$

Adding these two results, we get

$$2I=\lim_{n\to\infty}\sum_{r=1}^{n}\left(z_{r+}z_{r-1}\right)\left(z_r-z_{r-1}\right)$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\left(z_r^2-z_{r-1}^2\right)=\lim_{n\to\infty}\left(z_n^2-z_0^2\right).$$

Therefore, $I=\frac{1}{2}\left(\beta^2-\alpha^2\right)$.

In particular, if L is closed, then $\beta=\alpha$ and thus

$$\int\limits_L dz=0,\ \int\limits_L z\ dz=0.$$

2.1.8 Theorem (Integration along a Regular Arc): Let $f(z)$ be continuous on the regular arc L whose equation is $z(t)=x(t)+iy(t), t_0\le t\le T$. Prove that $f(z)$ is integrable along L and that

$$\int\limits_L f\left(z\right)\ dz=\int_{t_0}^{T}F\left(t\right)[x'(t)+i\,y'(t)]dt,$$

where F(t) denotes the value of $f(z)$ at the point of L corresponding to the parametric value t.

Proof: Let us consider the sum

$$\Sigma=\sum_{r-1}^{n}\ f(\xi_r)\ (z_r-z_{r-1})$$

where ξ_r is a point of L between z_{r-1} and z_r. If τ_r is the value of the parameter t corresponding to ξ_r, then τ_r lies between t_{r-1} and t_r. Writing $F(t)=\phi(t)+i\psi(t)$, where ϕ and ψ are real, we find that

$$\Sigma=\sum_{r=1}^{n}[\phi(\tau_r)\ +\ i\psi(\tau_r)]\left[(x_r-x_{r-1})\ +\ i(y_r-y_{r-1})\right]$$

$$=\sum_{r=1}^{n}\phi(\tau_r)(x_r-x_{r-1})\ +\ i\sum_{r=1}^{n}\phi(\tau_r)(y_r-y_{r-1})+i\sum_{r=1}^{n}\psi(\tau_r)(x_r-x_{r-1})\ -\ \sum_{r=1}^{n}\psi(\tau_r)(y_r-y_{r-1})$$

$$=\Sigma_1\ +i\Sigma_2\ +\ i\Sigma_3\ -\Sigma_4$$

$= (\Sigma_1 - \Sigma_4) + i\,(\Sigma_2 + \Sigma_3)$

We consider these four sums separately.

By the mean value theorem of differential calculus, the first sum is

$$\Sigma_1 = \sum_{r=1}^{n} \varphi(\tau_r)\,(x_r - x_{r-1})$$

$$= \sum_{r=1}^{n} \phi(\tau_r)\,x'(\tau_r')(t_r - t_{r-1})$$

Because, $(f(a+h) - f(a) = hf'(a+\theta h),\ 0 \le \theta \le 1.$

Therefore, $x_r - x_{r-1} = x(t_r) - x(t_{r-1})$

$=(t_r - t_{r-1})x'(\tau_r')$, where τ_r' lies between t_{r-1} and t_r.

We first show that Σ_1 can be made to differ by less than an arbitrary positive number, however small, from the sum $\Sigma_1' = \sum_{r=1}^{n} \phi(t_r)\,x'(t_r)(t_r - t_{r-1})$ by making the maximum of the differences $t_r - t_{r-1}$ sufficiently small.

Now, by hypothesis, the functions $\phi(t)$ and $x'(t)$ are continuous. As continuous functions are necessarily bounded, there exist a positive number K such that theinequalities $|\phi(t)| \le K, |x'(t)| \le K$ hold for $t_0 \le t \le T$.

Moreover, the functions are also uniformly continuous, we can, therefore, pre assign an arbitrary positive number $\in$, as small as we please, and then choose a positive number δ, depending on $\in$, such that

$$|\phi(t) - \phi(t')| < \in, |x'(t) - x'(t')| < \in, \text{whenever } |t - t'| < \delta.$$

Hence, if the maximum of the differences $t_r - t_{r-1}$ is less than δ, we have

$$|\phi(\tau_r)x'(\tau_r') - \phi(t_r)x'(t_r)| = |\phi(\tau_r)\{x'(\tau_r') - x'(t_r)\} + x'(t_r)\{\phi(\tau_r) - \phi(t_r)\}|$$

$$\le |\phi(\tau_r)|.|x'(\tau_r') - x'(t_r)| + |x'(t_r)|.|\phi(\tau_r) - \phi(t_r)|$$

$$< 2K\in$$

and therefore,

$$|\Sigma_1 - \Sigma_1'| < 2K\in (T - t_0).$$

Since we know that

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\delta x_i$$

By the definition of the integral of a continuous function of a real variable, Σ_1' tends to the limit

$$\int_{t_0}^{T} \phi(t)\,x'(t)dt$$

as $n\to\infty$ and the maximum of the differences $t_r - t_{r-1}$ tends to zero. Since $|\Sigma_1 - \Sigma_1'|$ can be made as small as we please by taking δ small enough, Σ_1 must also tend to the samelimit.

Similarly the other terms of Σ tend to limits. Combining these results, we find that Σ tends to the limit

$$\int_{t_0}^{T}[\phi(t)x'(t)-\psi(t)y'(t)]dt+i\int_{t_0}^{T}\psi(t)x'(t)-\phi(t)y'(t)dt$$

$$=\int_{t_0}^{T}F(t)[x'(t)+iy'(t)]dt$$

and so *f*(z) is integrable along the regular arc L.

2.1.9 Remark: The result of the theorem (2.1.8) is not merely of theoretical importance as an existence theorem. It is also of practical use since it reduces the problem of evaluating a complex integral to the integration of two real functions of a realvariable.

More generally, it can be shown that if *f*(z) is continuous on a contour C, it is integrable along C, the value of its integral being the sum of the integrals of *f*(z) along the regular arcs of which C is composed.

2.1.10 Example: Evaluate $\int\limits_L \frac{1}{z-a}dz$, where L is the circle $|z-a|=r$.

Solution: Let $I=\int\limits_L \frac{1}{z-a}dz$

The circle in the parametric form can be written as

$$z-a=re^{i\theta}, 0\le\theta\le 2\pi$$

$$\Rightarrow \quad z=a+re^{i\theta} \quad \Rightarrow dz=re^{i\theta}id\theta$$

Thus, $I=\int\limits_0^{2\pi}\frac{1}{re^{i\theta}}re^{i\theta}id\theta=\int\limits_0^{2\pi}id\theta=2\pi i$.

2.1.11 Theorem (Absolute Value of a Complex Integral): If *f*(z) is continuous on a contour C of length *l*, where it satisfies theinequality $|f(z)|\le M, then \left|\int\limits_C f(z)dz\right|\le M\,l$.

Proof: Without loss of generality, we assume that C is a regular arc.

Now, if g(t) is any complex continuous function of the real variable t, we have

$$\left|\sum_{r=1}^{n} g(t_r)(t_r-t_{r-1})\right|\le\sum_{r=1}^{n}|g(t_r)|(t_r-t_{r-1})$$

and so, on proceeding to the limit, we get

$$\left|\int_{t_0}^{T} g(t)dt\right|\le\int_{t_0}^{T}|g(t)|dt.$$

Hence, using the result of the previous theorem, we have

$$\left|\int_{t_0}^{T} f(z)\,dz\right| = \left|\int_{t_0}^{T} F(t)[x'(t)+iy'(t)]dt\right|$$

$$\leq \int_{t_0}^{T} |F(t)|\,|x'(t)+iy'(t)|\,dt$$

$$\leq M\int_{t_0}^{T} |x'(t)+iy'(t)|dt \qquad (\because f(z)=F(t)\,on\,C \Rightarrow |F(t)|\leq M)$$

$$= M\int_{t_0}^{T} \left|\frac{dz}{dt}\right|dt$$

$$= M\int_{t_0}^{T} |dz| = M\,l.$$

2.1.12 Remarks:

(i) The result of the above theorem (2.1.11) is also called estimate of the integral.

(ii) So far we had assumed that $f(z)$ is only continuous on the regular arc L along which we take its integral. We now impose the restriction that $f(z)$ is analytic and suppose further that L lies entirely within the simply connected domain D within which $f(z)$is regular. Then $\int_L f(z)\,dz$ certainly exists, since $f(z)$ is necessarily continuous on L. But we are no win a position to infer much more about this integral i.e. the integral is independent of path of integration. An equivalent form of this result is Cauchy theorem, the keystone in the theory of analytic functions.

First we consider the elementary form of Cauchy theorem which requires the additional assumption that the derivative of $f(z)$ is continuous. This form of Cauchy theorem is also known as **Cauchy fundamental theorem.**

2.1.13 Cauchy Theorem (Elementary Form): If $f(z)$ is analytic function whose derivative $f'(z)$ exists and is continuous at each point within and on a closed contour C,then $\int_C f(z)\,dz = 0$.

Proof: Let D denotes the closed region which consists of all points within and on C. If we write $z = x + iy, f(z) = u + iv$, then we have

$$\int_C f(z)dz = \int_C (u + iv)(dx + idy)$$

$$= \int_C (u\,dx - v dy) + i\int_C (v\,dx + u dy) \qquad (1)$$

Now, we use the Green's theorem for a plane which states that if $P(x, y), Q(x, y), \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ continuous functions within a domain Dare and if C is any closed contour in D, then

$$\int_C (P\,dx + Qdy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \qquad (2)$$

By hypothesis $f'(z)$ exists and is continuous in D, so u, v and their partial derivatives u_x, v_x, u_y, v_y are continuous functions of x and y in D. Thus the conditions of Green's theorem are satisfied. Hence applying this theorem in (1), weobtain

$$\int_C f(z)dz = \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\,dy + \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx\,dy \text{ (Using C-R equations)}$$

$$= \iint_D \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y} \right) dx\,dy + i \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x} \right) dx\,dy = 0 + i0 = 0.$$ This completes the result.

An important step was pointed out by **Goursat** who showed that it is unnecessary to assume the continuity of $f'(z)$, and that Cauchy's theorem is true if it is only assumed that $f'(z)$ exists at each point within and on C. Actually, the continuity of the derivative $f'(z)$ and its differentiability are consequences of Cauchy's theorem. This form of Cauchy theorem is also known as **Cauchy-Goursat Theorem.**

2.1.14 General Form of Cauchy's Theorem: If a function $f(z)$ is analytic and one-valued within and on a simple closed contour C, then $\int_C f(z)dz = 0$.

Proof: First of all, we observe that the integral certainly exists, since a function which is analytic is continuous and a continuous function is integrable. For the proof of the theorem, we divide up the region inside the closed contour C into a large number of sub-regions by a network of lines paralleltotherealandimaginaryaxes.SupposethatthisdividestheinsideofCintoanumberof squares $C_1, C_2, \ldots C_M$ say, and a number of irregular regions $D_1, D_2, \ldots, D_N$ say, parts of whose boundaries are parts of C (Fig. 1) .

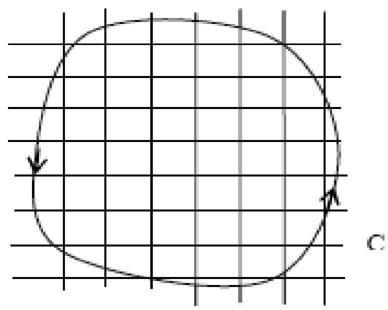


Fig. 1

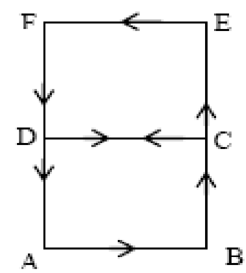


Fig. 2

Then

$$\int_C f(z)dz = \sum_{m=1}^{M} \int_{C_m} f(z)dz + \sum_{n=1}^{N} \int_{D_n} f(z)dz , \tag{1}$$

where each contour is described in positive (anti-clockwise) direction.

Consider, for example, any two adjacent squares ABCD and DCEF with common side CD (Fig.2). The side CD is described from C to D in the first square and from D to C in the second. Hence, the two integrals along CD cancel. So all the integrals cancel except those which form part of C itself, since these are described once only. Moreover within the integrals of R.H.S. of (1), there are contained integral along all the parts of the contour C into which C is divided on account of the subdivision. Thus the result (1) is true.

We now use the fact that $f(z)$ is analytic at every point. This means that, if z_0 is any point inside or on C,then

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon, \text{ provided that } 0 < |z - z_0| < \delta = \delta(z_0)$$

i.e. if $|z-z_0| < \delta$, then $|f(z) - f(z_0) - (z-z_0) f'(z_0)| \leq \in |z-z_0|$ (2)

If we consider any particular region C_m or D_n in the above construction, it is evident that we can choose its side so small that (2) is satisfied if z_0 is a given point of the region, and z is any other point. It is not, however, immediately obvious that we can choose the whole network so that the conditions are satisfied in all the partial regions at the same time. We shall prove that this is actually possible. i.e. "having given $\in$, we can choose the network in such a way that, in every C_m or D_n , there is a point z_0 such that (2) holds for every z in this region". This actually, means that the function is uniformly differentiable throughout the interior of C. We prove it by well known process of sub division.

Suppose that we start with a network of parallel lies at constant distance l between every consecutive pair of lines. Some of the squares formed by these lines may each contain a point $z_0$of the desired type. We leave these squares as they are. The rest we subdivide by lines midway between the previous lines. If there still remain any parts which do not have the required property, we subdivide them again in the same way. Obviously, there are two distinct possibilities. The process may terminate after a finite number of steps and then the result is obtained, or it may go on in definitely.

In the second case, there is at least one region which we can subdivide indefinitely without obtaining the required result. We call this region, including its boundaries, R_1. After the first sub division, we obtain a part R_2 contained in R_1 with the same property. Proceeding in this way, we have an infinity of regions $R_1, R_2, \ldots, R_n$ each contained in the previous one, and in each of which inequality (2) is impossible. Since $R_1 \supset R_2 \supset R_3, \ldots$, there must be a point z_0 common to all the regions R_n ($n = 1, 2, \ldots$) and since the dimensions of R_n decrease indefinitely, we can have $|z-z_0| < \delta$ for sufficiently large n, say $n > n_0$ and for every z in R_n. But $f(z)$ is analytic at z_0. Hence (2) holds for this z_0 in R_n if $n > n_0$. This contradicts the statement that in no R_n, there exists a point $z_0$satisfying inequality (2). Thus the second possibility is ruled out and (2) is satisfied for every point in the region C.

Now, let us consider one of the squares C_m of side l_m. In C_m, by inequality (2), we have

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + \phi(z), \text{ where } |\phi(z)| \leq \in |z - z_0|$$

Hence, $$\int_{C_m} f(z)\,dz = \int_{C_m} [f(z_0) + (z-z_0) f'(z_0)]\,dz + \int_{C_m} \phi(z)\,dz \quad (3)$$

The first integral in (3) simplifies to

$[f(z_0) - z_0 f'(z_0)] \int_{C_m} dz + f'(z_0) \int_{C_m} z\,dz$ and therefore vanishes, since $\int_{C_m} dz = 0$, $\int_{C_m} z\,dz = 0$ (by definition).

Also, by virtue of the result regarding absolute value of a complex integral, we obtain

$$\left|\int_{C_m} \phi(z)\,dz\right| < \in \int_{C_m} |z - z_0||dz|$$
$$< \in \sqrt{2} l_m . 4 l_m ,$$

since $|z-z_0| \leq \sqrt{2}\, l_m$ for z_0 inside C_m and z on C_m and the length of C_m is $4l_m$

In the case of any one of the irregular region D_n, the length of the contour is not greater than $4l_n + \delta_n$, where δ_n is the length of the curved part of the boundary. Hence

$$\left| \int_{D_n} \phi(z)dz \right| < \in \sqrt{2} l_n (4l_n + \delta_n).$$

Adding all the parts, we obtain

$$\left| \int_C f(z)dz \right| < \sum_{m=1}^{M} \left| \int_{C_m} f(z)dz \right| + \sum_{n=1}^{N} \left| \int_{D_n} f(z)dz \right|$$

$$= \sum_{m=1}^{M} \left| \int_{C_m} \phi(z)dz \right| + \sum_{n=1}^{N} \left| \int_{D_n} \phi(z)dz \right|$$

$$< \sum \in \sqrt{2}\, 4l_m^2 + \sum \in \sqrt{2} l_n (4l_n + \delta_n)$$

$$< 4\sqrt{2} \in \sum (l_m^2 + l_n^2) + \in \sqrt{2}\, l \sum \delta_n \qquad (4)$$

where l denotes some constant greater than every one of the $l_n{}'s$. Now $\sum \left(l_m{}^2 + l_n{}^2 \right)$ is the area of a region which just includes C and is therefore bounded. Also $\sum \delta_n$ is the length of the contour C. Hence the R.H.S. of (4) is less than a constant multiple of $\in$. But the L.H.S. is independent of $\in$, and $\in$ is arbitrarily small, it follows therefore that $\int_C f(z)dz = 0$ which proves the theorem.

2.1.15 Corollary: Suppose *f*(z) is analytic in a simply connected domain D, then the integral along any rectifiable curve in D joining any two points of D is the same i.e. it does not depend on the curve joining the two points i.e. integral is independent of path.

Proof: Suppose the two points $A(z_1)$ and $B(z_2)$ of the simply connected domain D are joined by the curves C_1 and C_2 as shown in the figure.

Then, by Cauchy's theorem

$$\int_{ALBMA} f(z)dz = 0$$

i.e. $\int_{ALB} f(z)dz + \int_{BMA} f(z)dz = 0$

i.e. $\int_{ALB} f(z)dz - \int_{AMB} f(z)dz = 0$

i.e. $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$

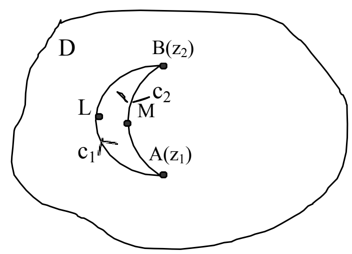


Figure 5

2.1.16 Extension of Cauchy's Theorem to Contours Defining Multiply Connected Region: By adopting a suitable convention as to the sense of integration, Cauchy's theorem can be extended to the case of contours which are made up of several distinct closed contours. Consider, for example, a function *f*(z) which is analytic in the multiply connected region R bounded by the closed contour C and the two interior contours C_1, C_2 as well as on these contours themselves. The complete contour C^* which is the boundary of the region R is made up of the three contours C, C_1 and C_2 and we adopt the convention that C^* is described in the positive sense if the region R is on the L.H.S. w.r.t. this sense of describing it. Then by Cauchy's theorem

$$\int_{C^*} f(z)\, dz = 0$$

where the integral is taken round the complete contour C^* in the positive sense.

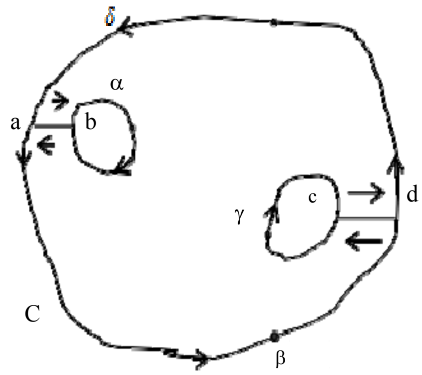


Figure 6

Practically, we deal with this case by drawing transversals like ab, cd and by applying Cauchy's theorem for a simple closed contour abαbaβdcγcdδa. It is found convenient in applications to express the same result in the form

$$\int_{C} f(z)\, dz = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz$$

where all the three integrals are now taken in the same (positive) sense.

An exactly similar result holds in case there are any finite number of closed contours $C_1, C_2, \ldots, C_m$ inside a closed contour C and $f(z)$ is analytic in the multiply connected region bounded by them as well as on them. Then we have

$$\int_{C} f(z)\, dz = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz + \ldots + \int_{C_m} f(z)\, dz$$

whereall the contours are described in positive sense.

2.1.17Theorem (Cauchy's Integral Formula): Let $f(z)$ be analytic inside and on a closed contour C and let z_0 be any point inside C.Then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\, dz.$$

Proof: We consider the function $\frac{f(z)}{z - z_0}$. This function is analytic throughout the region bounded by C except at $z - z_0$. Then, by 2.1.16, We have

$$\int_C \frac{f(z)}{z-z_0}dz = \int_\gamma \frac{f(z)}{z-z_0}dz$$

where γ is any closed contour inside C including the point z_0 as an interior point.

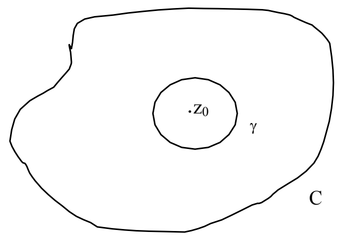


Figure 7

Let us choose γ to be the circle with centre z_0 and radius ρ. Since *f*(z) is continuous, we can take ρ so small that on γ, $|f(z)-f(z_0)|<\in$ where $\in$ is any pre assigned positive number.

Now, $$\int_\gamma \frac{f(z)}{z-z_0}dz = \int_\gamma \frac{[f(z)-f(z_0)]+f(z_0)}{z-z_0}dz$$

$$= f(z_0)\int_\gamma \frac{1}{z-z_0}dz + \int_\gamma \frac{f(z)-f(z_0)}{z-z_0}dz \qquad (1)$$

For any point z on γ,

$$z-z_0 = \rho e^{i\theta} \Rightarrow dz = \rho i e^{i\theta}d\theta$$

$$\int_\gamma \frac{1}{z-z_0}dz = \int_0^{2\pi} \frac{\rho e^{i\theta}d\theta}{\rho e^{i\theta}} = \int_0^{2\pi} i d\theta = 2\pi i$$

and

$$\left|\int_\gamma \frac{f(z)-f(z_0)}{z-z_0}dz\right| = \left|\int_0^{2\pi} \frac{f(z)-f(z_0)}{\rho e^{i\theta}}\rho e^{i\theta} i d\theta\right|$$

$$= \left|\int_0^{2\pi}[f(z)-f(z_0)]id\theta\right|$$

$$<\in \int_0^{2\pi} d\theta = 2\pi \in$$

Hence from (1), we get

$$\left|\int_C \frac{f(z)}{z-z_0}dz - 2\pi i f(z_0)\right| < 2\pi \in .$$

Since $\in$ is arbitrarily small and L.H.S. is independent of $\in$, it follows that

$$\int_C \frac{f(z)}{z-z_0}dz - 2\pi i f(z_0) = 0$$

Therefore, $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz$.

Which proves the result.

2.1.18 Corollary(Extension of Cauchy's Integral Formula to Multiply Connected Region): If $f(z)$ is analytic in a ring shaped region bounded by two closed contours C_1 and C_2 and z_0 is a point in the region between C_1 and C_2, then

$$f(z_0) = \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{z-z_0}dz - \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-z_0}dz,$$ where C_2 is the outer contour.

Proof: Describe a circle γ of radius ρ about the point z_0 such that the circle lies in the ring shaped region. The function $\frac{f(z)}{z-z_0}$ is analytic in the region bounded by three closed contours C_1, C_2 and γ

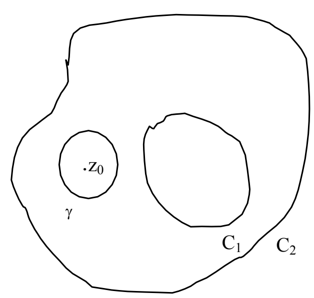


Figure 8

Thus by 2.1.16, we have

$$\int_{C_2} \frac{f(z)}{z-z_0}dz = \int_{C_1} \frac{f(z)}{z-z_0}dz + \int_{\gamma} \frac{f(z)}{z-z_0}dz$$

where the integral along each contour is taken in positive sense. Now, using Cauchy's integral formula, we find

$$\int_{C_2} \frac{f(z)}{z-z_0}dz = \int_{C_1} \frac{f(z)}{z-z_0}dz + 2\pi i f(z_0)$$

Hence, $f(z_0) = \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{z-z_0}dz - \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-z_0}dz.$ which proves the result.

2.1.19 Poisson's Integral Formula: Let f(z) be analytic in the region $|z| \le R$ then for $0 < r < R,$ we have

$$f(r\,e^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2) f(\mathrm{Re}^{i\phi})}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\, d\phi$$

where ϕ is the value of θ on the circle $|z| = R$.

Proof: Let C denote the circle $|z| = R$.Let $z_0 = r\,e^{i\theta}, \theta < r < R$ by any point inside C, then by Cauchy's integral formula,

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\, dz \qquad (1)$$

The inverse of z_0 w.r.t. the circle $|z| = R$ is $\dfrac{R^2}{\overline{z_0}}$ and lies outside the circle, so by Cauchy's theorem, we have

$$0 = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - \dfrac{R^2}{\overline{z_0}}}\, dz \qquad (2)$$

Subtracting (2) from (1), we get

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{(z_0 - \dfrac{R^2}{\overline{z_0}}) f(z) dz}{(z - z_0)(z - \dfrac{R^2}{\overline{z_0}})}$$

$$= \frac{1}{2\pi i}\int_C \frac{(R^2 - z_0\overline{z_0}) f(z) dz}{(z - z_0)(R^2 - z\overline{z_0})} \qquad (3)$$

Now, any point of circle C is expressible as $z = R\,e^{i\phi}$. Also $z_0 = r\,e^{i\theta}$, so $\overline{z_0} = r\,e^{-i\theta}$.Therefore,

$$R^2 - z_0\overline{z_0} = R^2 - r^2 \qquad (4)$$

$$(z - z_0)(R^2 - z\overline{z_0}) = zR^2 - z^2\overline{z_0} - z_0R^2 + z_0\overline{z_0}z$$

$$= R^3e^{i\phi} - R^2e^{i2\phi}r\,e^{-i\theta} - r\,e^{i\theta}R^2 + r^2R\,e^{i\phi}$$

$$= R\,e^{i\phi}[R^2 - 2r\,R\cos(\theta - \phi) + r^2] \qquad (5)$$

and $dz = R\,i\,e^{i\phi}d\phi$.

Thus, (3) becomes

$$f(re^{i\theta})=\frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2-r^2)f(Re^{i\phi})d\phi}{R^2-2Rr\cos(\theta-\phi)+r^2} \tag{6}$$

which is the required result.

Formula (6) can be separated into real and imaginary parts to get (f(z)=u+iv)

$$u(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2-r^2)u(R,\phi)d\phi}{R^2-2Rr\cos(\theta-\phi)+r^2}$$

$$v(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2-r^2)v(R,\phi)d\phi}{R^2-2Rr\cos(\theta-\phi)+r^2}.$$

2.1.20 Cauchy's Integral Formula for the Derivative of an Analytic Function: If a function f(z) is analytic within and on a simple closed contour C and z_0 is any point inside C, then

$$f'(z_0)=\frac{1}{2\pi i}\int_C\frac{f(z)}{(z-z_0)^2}dz.$$

Proof: Let z_0+h be a point in the neighbourhood of point $z_0$and inside C. Then, Cauchy's integral formula at these two points gives

$$f(z_0)=\frac{1}{2\pi i}\int_C\frac{f(z)}{z-z_0}dz$$

and $$f(z_0+h)=\frac{1}{2\pi i}\int_C\frac{f(z)}{z-(z_0+h)}dz$$

$$\frac{f(z_0+h)-f(z_0)}{h}=\frac{1}{2\pi i}\int_C\frac{f(z)}{h}\left[\frac{1}{z-(z_0+h)}-\frac{1}{z-z_0}\right]dz$$

$$=\frac{1}{2\pi i}\int_C\frac{f(z)}{(z-z_0)h}\left[\frac{z-z_0}{z-(z_0+h)}-1\right]dz$$

$$=\frac{1}{2\pi i}\int_C\frac{f(z)}{(z-z_0)h}\left[\left(\frac{z-z_0-h}{z-z_0}\right)^{-1}-1\right]dz$$

$$=\frac{1}{2\pi i}\int_C\frac{f(z)}{(z-z_0)h}\left[\left(1-\frac{h}{z-z_0}\right)^{-1}-1\right]dz$$

$$=\frac{1}{2\pi i}\int_C\frac{f(z)}{(z-z_0)h}\left[1+\frac{h}{z-z_0}+\left(\frac{h}{z-z_0}\right)^2+\ldots\ldots-1\right]dz$$

$$= \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)} \left[\frac{1}{z-z_0} + \frac{h}{(z-z_0)^2} + \ldots\ldots \right] dz .$$

Taking limit as h→0, we have

$$\lim_{h \to 0} \frac{f(z_0+h)-f(z_0)}{h} = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)} \left[\frac{1}{z-z_0} + 0 + \ldots\ldots \right] dz .$$

Hence, *f(z)* is differentiable at z_0 and

$$f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^2} dz .$$

2.1.21 Generalization: This result (2.1.20) has a very significant consequence in the fact that $f'(z)$ is itself analytic within C. i.e. derivative of an analytic function is also analytic. To prove this, it is enough to show that $f'(z)$ has derivative at any point z_0 inside C. Using Cauchy's integral formula for $f'(z_0)$ and $f'(z_0+h)$ with the same restriction on h as before, we get

$$f'(z_0+h) - f'(z_0) = \frac{1}{2\pi i} \int_C f(z) \left[\frac{1}{(z-z_0-h)^2} - \frac{1}{(z-z_0)^2} \right] dz$$

or $$\frac{f'(z_0+h) - f'(z_0)}{h} = \frac{1}{2\pi i} \int_C \frac{f(z)(2z-2z_0-h)}{(z-z_0)^2(z-z_0-h)^2} dz$$

by means of arguments parallel to those used in the proof of Cauchy's formula for $f'(z_0)$, we can easily show that ash→0, the integral on R.H.S. tends to the limit

$$\frac{2}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^3} dz$$

Thus $f'(z)$ has a differential co-efficient at z_0, given by the formula

$$f''(z_0) = \frac{\lfloor 2}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^3} dz$$

The arguments can obviously be repeated and we get the following result as a generalization." If *f*(z) is analytic inside and on a closed contour C, it possesses derivatives of all orders which are all analytic inside C. The nth derivative $f^n(z_0)$ at any point z_0 inside C being given by the formula

$$f^n(z_0) = \frac{\lfloor n}{2\pi i} \int_C \frac{f(z)dz}{(z-z_0)^{n+1}} ."$$

2.1.22 Remark: From Cauchy integral formula, we observe a remarkable fact about an analytic function. Its values everywhere inside a closed contour are completely determined by its values on the boundary. In fact the values of each derivative of an analytic function are determined just by the values of the function on the boundary.

2.1.23 Example: Evaluate $\int_C \frac{dz}{(z-z_0)^m}, m = 1, 2, \ldots M$ where C is a single closed contour.

Solution: The function $\frac{1}{(z-z_0)^m}$ is analytic except at z = z_0. Hence if C does not enclose z_0, then by Cauchy's theorem, the integral is zero. If C encloses z_0, then we choose a circle γ of small radius ρ with centre z_0.

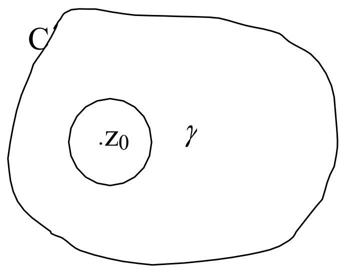


Figure 9

Thus, we get

$$I = \int_C \frac{dz}{(z-z_0)^m} = \int_\gamma \frac{dz}{(z-z_0)^m},$$

On $\gamma, z - z_0 = \rho e^{i\theta} id\theta$,

$$\therefore \qquad I = \int_0^{2x} \frac{\rho e^{i\theta} id\theta}{\rho^m e^{im\theta}} = \int_0^{2x} i\rho^{-m+1} e^{-i(m-1)\theta} d\theta = \begin{cases} 2\pi i, & if\ m = 1 \\ 0 \ , & if\ m \neq 1 \end{cases}$$

$$\text{Therefore, } I = \begin{cases} 0, if\ z = z_0\ is\ outside\ C \\ 0, if\ z = z_0\ is\ inside\ C, m \neq 1 \\ 2\pi i\ if\ z = z_0\ is\ inside\ C, m = 1. \end{cases}$$

2.1.24 Example: Evaluate $\int_c \frac{e^z}{z(z^2-16)} dz$, where C is a closed contour between the circles of radius 1 and 3, centred at origin.

Solution: The integrand is analytic except at z = 0, z = $\pm$ 4 which are not points of the given region. Therefore, by Cauchy's theorem, the integral vanishes.

2.1.25 Example: Using Cauchy`s Integral formula show that

$$\int_C \frac{e^{2z}}{(z+1)^4} dz = \frac{8\pi e^{-2}}{3} i \text{, where C is the circle } |z| = 3.$$

Solution: By Cauchy's integral formula for derivatives, we have

$$f^n(z_0) = \frac{\underline{|n}}{2\pi i} \int_C \frac{f(z)dz}{(z-z_0)^{n+1}}$$

where $f(z)$ is analytic inside and on C.

In the present case, C is $|z| = 3$,

$f(z) = e^{2z}$, $z_0 = -1$, $n = 3$ and $f(z)$ is analytic inside and on the circle $|z| = 3$.

Also, $f^3(-1) = 8e^{-2}$.Therefore, (1) becomes

$$8e^{-2} = \frac{\lfloor 3}{2\pi i} \int_C \frac{e^{2z}}{(z+1)^4} dz$$

$$\Rightarrow \qquad \int_C \frac{e^{2z}}{(z+1)^4} dz = \frac{8\pi e^{-2}}{3} i$$

Hence the result.

2.1.26 Example: Using Cauchy Integral formula, find the value of the integral

$\int_C \frac{dz}{z(z+i\pi)}$ where C is a circle given by $|z+3i| = 4$.

Solution: By using partial fraction, we have

$$\int_C \frac{dz}{z(z+i\pi)} = \frac{1}{\pi i} \int_C \left(\frac{1}{z} - \frac{1}{z+i\pi}\right) dz$$

$$= \frac{1}{\pi i} \int_C \frac{dz}{z} - \int_C \frac{dz}{(z+i\pi)}$$

$$= \frac{1}{\pi i}(I_1 - I_2) \qquad (*)$$

Now, $I_1 = \int_C \frac{dz}{z}$. Here $f(z) = 1, a = 0$.

By using Cauchy integral formula, $f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$; $f(0) = \frac{1}{2\pi i} \int_C \frac{1}{z} dz$

$$1 = \frac{1}{2\pi i} \int_C \frac{1}{z} dz \Rightarrow \int_C \frac{1}{z} dz = 2\pi i$$

Now, $I_2 = \int_C \frac{dz}{(z+i\pi)}$ here $f(z) = 1, a = -i\pi$.

By using Cauchy integral formula, $f(-i\pi) = \frac{1}{2\pi i} \int_C \frac{1}{z+i\pi} dz$

$$1 = \frac{1}{2\pi i} \int_C \frac{1}{z+i\pi} dz \Rightarrow 2\pi i = \int_C \frac{1}{z+i\pi} dz$$

Putting the value of I_1 and I_2 in (*), we get

$$\int_C \frac{dz}{z(z+i\pi)} = \frac{1}{\pi i}\int (2\pi i - 2\pi i)dz = 0.$$

2.1.27 Exercise: Using Cauchy's Integral formula, prove that

$(i)\ \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = 4\pi i$, where C is the circle |z|=3

$(ii)\ \int_C \frac{\sin^6 z}{(z-\pi/6)^3} dz = \frac{21}{16}\pi i$, where C is the circle |z|=1.

$(iii)\ \int_C \frac{\cos z}{z(z-4)} dz = \frac{-\pi i}{2}$, where C is the circle |z|=1.

$(iv)\ \int_C \frac{e^{zt}}{(z^2+1)^2} dz = \frac{1}{2}(\sin t - t\cos t)$, where t > 0 and C is the circle |z|=3.

2.1.28 A Complex Integral as a Function of its Upper Limit: Let *f*(z) be analytic in a region D and let

$$F(z) = \int_{z_0}^{z} f(w)\,dw$$

where z_0 is any fixed point in D and the path of integration is any contour from z_0 to z lying entirely in D. It follows from Cor. (2.1.15) to Cauchy's theorem that the value of F(z) depends on z only and not on the particular path of integration from z_0 to z. F(z) is called the indefinite integral of *f*(z). We prove below the analogue, in the theory of functions of a complex variable, of the well known "fundamental theorem of integral calculus". It asserts that the operations of integration and differentiation are inverse operations.

2.1.29 Theorem: The function F(z) is analytic in D and its derivative is f(z).

Proof: Since $F(z) = \int_{z_0}^{z} f(w)\,dw$

and $F(z+h) = \int_{z_0}^{z+h} f(w)\,dw$

Thus, $F(z+h) - F(z) = \int_{z_0}^{z+h} f(w)\,dw - \int_{z_0}^{z} f(w)\,dw$

$$= \int_{z}^{z_0} f(w)\,dw + \int_{z_0}^{z+h} f(w)\,dw$$

$$= \int_{z}^{z+h} f(w)\,dw$$

Hence, $\frac{F(z+h)-F(z)}{h} = \frac{1}{h}\int_{z}^{z+h} f(w)\,dw.$

By Cauchy's theorem, we may suppose that integral is taken along the straight line from z to z+h. Thus

$$\frac{F(z+h)-F(z)}{h}-f(z)=\frac{1}{h}\int_{z}^{z+h} f(w)\,dw-\frac{1}{h}\int_{z}^{z+h} f(z)\,dw$$

$$=\frac{1}{h}\int_{z}^{z+h} [f(w)-f(z)]dw$$

Since f(z) is analytic so it is continuous, given $\in > 0$, there exists a $\delta > 0$ such that

$$\left|f(w)-f(z)\right| < \in \text{ whenever } |w-z| < \delta.$$

Therefore, if $0 < |h| < \delta$, we have

$$\left|\frac{F(z+h)-F(z)}{h}-f(z)\right| \le \frac{1}{|h|}\int_{z}^{z+h} |f(w)-f(z)||dw|$$

$$< \frac{1}{|h|}\int_{z}^{z+h} \in |dw| = \frac{1}{|h|} \in |h| = \in$$

Hence, $\lim\limits_{h\to 0} \dfrac{F(z+h)-F(z)}{h}-f(z)=0$

or $\quad F'(z)=f(z)$

which proves that F(z) is analytic and that its derivative is f(z).

2.1.30 Morera's Theorem (Converse of Cauchy's Theorem): If *f*(z) is continuous in a region D and if the integral ∫*f*(z)dz taken round any closed contour in D vanishes, then *f*(z) is analytic in D.

Proof: When the integral round a closed contour vanishes, then we know that the value of the integral

$$F(z)=\int_{z_0}^{z} f(w)\,dw$$

is independent of path of integration joining z_0 and z. Also, we have

$$\frac{F(z+h)-F(z)}{h}=\frac{1}{h}\int_{z}^{z+h} f(w)dw$$

and further

$$\frac{F(z+h)-F(z)}{h}-f(z)=\frac{1}{h}\int_{z}^{z+h} [f(w)-f(z)]dw$$

where we are free to assume that the path of integration is the straight line joining the points z and z+h. Since f(z) is continuous in D, we find that (previous theorem 2.1.29)

$$F'(z)=f(z)$$

i.e. F(z) is analytic with derivative *f(z)*. But we have the result that derivative of an analytic function is analytic. Thus, we finally conclude that $F'(z)$ i.e. *f(z)* is analytic in D.

2.1.31 Example: Find the value of the integral $\int (x-y+ix^2)dz$

(i) along the straight line from z = 0 to z = 1+i

(ii) along the real axis from z = 0 to z = 1 and then along a line parallel to imaginary axis from z = 1 to z = 1+i.

Solution: Let $z = x + iy \Rightarrow dz = dx + idy$

(i) OA is the straight line joining z = 0 to z = 1+i.

Clearly y = x on A

$\therefore dy = dx$

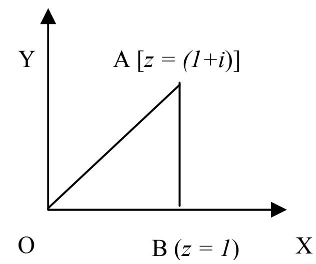


Figure 10

$$\int_{OA} (x - y + ix^2)dz = \int_0^1 (x - x + ix^2)(dx + idx)$$

$$= i(1+i)\int_0^1 x^2 dx = \frac{i(1+i)}{3} = \frac{(i-1)}{3}.$$

(ii) The real axis from z = 0 to z = 1 is the line OB and y = 0 on OB.

Therefore, z = x,dz = dx

$$\int_{OB} (x - y + ix^2)dz = \int_0^1 (x - 0 + ix^2)dx = \int_0^1 (x + ix^2)dx = \frac{1}{2} + \frac{i}{3}$$

Now, BA is the line parallel to the imaginary axis from z = 1 to z = 1+i and x = 1 on BA so that dx = 0, dz = idy

$$\therefore \int_{BA} (x - y + ix^2)dz = \int_0^1 (1 - y + i)idy = [(1+i) - \frac{1}{2}]i = \frac{i}{2} - 1 .$$

2.1.32 Cauchy's Inequality (Cauchy's Estimate): If f(z) is analytic within and on a circle C given by $|z - z_0| = R$ and if $|f(z)| \le M$ for every z on C, then $\left|f^n(z_0)\right| \le \frac{M\lfloor n}{R^n}$.

Proof: Since *f*(z) is analytic inside C, we have by Cauchy's integral formula for nth derivative of an analytic function

$$f^n(z_0) = \frac{\lfloor n}{2\pi i}\int_c \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

Since on the circle $|z - z_0| = R,\ z - z_0 = \mathrm{Re}^{i\theta}, dz = \mathrm{Re}^{i\theta}\, id\theta$ and the length of the circle is $2\pi R$, therefore

$$\left|f^n(z_0)\right| = \frac{\lfloor n}{2\pi}\left|\int_C \frac{f(z)dz}{(z - z_0)^{n+1}}\right|$$

$$\leq \frac{\lfloor n}{2\pi}\int_C \frac{|f(z)||dz|}{|z-z_0|^{n+1}}$$

$$\leq \frac{\lfloor n}{2\pi}\int_0^{2\pi} \frac{M\left|\mathrm{Re}^{i\theta}\, id\theta\right|}{\left|\mathrm{Re}^{i\theta}\right|^{n+1}} = \frac{\lfloor n}{2\pi}\int_0^{2\pi}\frac{M}{R^n}d\theta$$

$$= \frac{\lfloor n}{2\pi}\frac{M}{R^n}2\pi = \frac{M\lfloor n}{R^n}$$

Hence, $\left|f^n(z_0)\right| \leq \frac{M\lfloor n}{R^n}$.

2.1.33 Liouville's Theorem: A function which is analytic in all finite regions of the complex plane, and is bounded, is identically equal to a constant .i.e. the only bounded entire functions are the constant functions.

Proof: Let z_1, $z_2$be arbitrary distinct points in z-plane and let C be a large circle with centre at origin and radius R such that C enclosesz$_1$ andz$_2$i.e. $|z_1| < R, |z_2| < R$.

Since *f*(z) is bounded, there exists a positive number M such that$|f(z)| \leq M \forall z$.

By Cauchy's integral formula,

$$f(z_1) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-z_1}$$

$$f(z_2) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-z_2}$$

$$f(z_2) - f(z_1) = \frac{1}{2\pi i}\int_C \frac{f(z)(z_2-z_1)}{(z-z_2)(z-z_1)}dz$$

Thus, $$f(z_2)-f(z_1)\ \ | \leq \frac{|z_2-z_1|}{2\pi}\int_C \frac{|\ f(z)\ ||dz|}{|z-z_1||z-z_2|}$$

$$\leq \frac{M|z_2-z_1|}{2\pi}\int_C \frac{|dz|}{|z-z_1||z-z_2|}$$

$$\leq \frac{M|z_2-z_1|}{2\pi}\int_C \frac{|dz|}{(|z|-|z_1|)(|z|-|z_2|)} \qquad \left|\because |z-z_1| \geq |z|-|z_1|\right|$$

Now, on the circle C, $z = \mathrm{Re}^{i\theta}, |z| = R,\ dz = \mathrm{Re}^{i\theta}\, id\theta$

Therefore, $$|f(z_2)-f(z_1)| \leq \frac{M|z_2-z_1|}{2\pi}\int_0^{2x}\frac{|\mathrm{Re}^{i\theta}\, id\theta|}{(R-|z_1|)(R-|z_2|)}$$

$$= \frac{M|z_2 - z_1|}{2\pi} \frac{R}{(R-|z_1|)(R-|z_2|)} 2\pi$$

$$= \frac{M|z_2 - z_1|}{\left(1-\frac{|z_1|}{R}\right)\left(1-\frac{|z_2|}{R}\right)} \frac{1}{R}$$

which tends to zero as R → ∞. Hence, $f(z_2) - f(z_1) = 0$ i.e. $f(z_1) = f(z_2)$. But z_1, z_2 are arbitrary, this holds for all couples of points z_1, z_2 in the z-plane, therefore *f*(z)=constant.

2.1.34 Taylor's Series: We have observed that a convergent complex power series defines an analytic (holomorphic) function. Here, we discuss its converse i.e. we proceed to prove that if *f*(z) is an analytic function, regular in a neighbourhood of the point z = a, it can be expanded in a series of powers of (z –a). These two results combine to demonstrate that a function is analytic in a region iff it is locally representable by power series. The following theorem extends Taylor's classical theorem in real analysis to analytic functions of a complex variable.

2.1.35 Taylor's Theorem: Suppose that *f*(z) is analytic inside and on a closed contour C and let a be a point inside C. Then

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{\lfloor 2}(z-a)^2 + ... + \frac{f^n(a)}{\lfloor n}(z-a)^n$$

$$= f(a) + \sum_{n=1}^{\infty} \frac{f^n(a)}{\lfloor n}(z-a)^n$$

The infinite series is convergent if $|z-a| < \delta$ where δ is the distance from a to the nearest point ofC. In the region $|z-a| \le \delta_1$ where $\delta_1 < \delta$, the series is uniformly covergent.

Proof: Let $\delta_2 = \frac{\delta + \delta_1}{2}$ *so that* $0 < \delta_1 < \delta_2 < \delta$. Then, by hypothesis, *f*(z) is analytic within and on the circle γ defined by the equation $|z-a| = \delta_2$. Let a + h be any point of the region defined by $|z-a| \le \delta_1$.

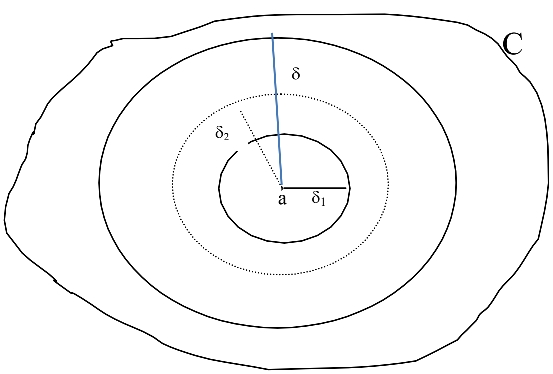


Figure 11

Since a + h lies within the circle γ, using Cauchy`s integral formula

$$f(a+h)=\frac{1}{2\pi i}\int_{\gamma}\frac{f(z)}{z-a-h}dz$$

$$=\frac{1}{2\pi i}\int_{\gamma}f(z)\frac{1}{(z-a)\left(1-\frac{h}{z-a}\right)}dz=\frac{1}{2\pi i}\int_{\gamma}\frac{f(z)}{z-a}\left[\frac{1}{1-\frac{h}{z-a}}\right]dz$$

$$=\frac{1}{2\pi i}\int_{\gamma}\frac{f(z)}{z-a}\left[1+\frac{h}{z-a}+\frac{h^2}{(z-a)^2}+\ldots\ldots\frac{h^n}{(z-a)^n}+\frac{h^{n+1}}{(z-a)^n(z-a-h)}\right]dz$$

$$\left(\because\frac{1}{1-b}=1+b+b^2+\ldots.+b^n+\frac{b^{n+1}}{1-b}\right)$$

$$=\frac{1}{2\pi i}\int_{\gamma}\frac{f(z)}{z-a}dz+\frac{h}{2\pi i}\int_{\gamma}\frac{f(z)}{(z-a)^2}dz+\frac{h^2}{2\pi i}\int_{\gamma}\frac{f(z)}{(z-a)^3}dz$$

$$+\ldots.+\frac{h^n}{2\pi i}\int_{\gamma}\frac{f(z)}{(z-a)^{n+1}}dz+\frac{h^{n+1}}{2\pi i}\int_{\gamma}\frac{f(z)dz}{(z-a)^{n+1}(z-a-h)}$$

Using Cauchy's integral formulae for the derivatives of an analytic function, we get

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{\lfloor 2}f''(a)+\ldots..+\frac{h^n}{\lfloor n}f^n(a)+\Delta_n$$

where $\Delta_n=\dfrac{h^{n+1}}{2\pi i}\displaystyle\int_{\gamma}\frac{f(z)dz}{(z-a)^{n+1}(z-a-h)}$.

Thus, $\quad f(a+h)=f(a)+\sum_{r=1}^{n}f^r(a)\dfrac{h^r}{\lfloor r}+\Delta_n$

But on account of continuity, *f*(z) is bounded on the circle γ. Thus there exists a positive constant M such that $|f(z)|\le M$ on γ. Also, when $|z-a|=\delta_2$,

$$|z-a-h|\ge|z-a|-|h|\ge\delta_2-\delta_1$$

where a+h lies in the circle and $|z-a|=\delta_1$, implies $|a+h-a|\le\delta_1$ *i.e.* $|h|\le\delta_1$.

Now, applying the result regarding the absolute value of a complex integral we have the inequality

$$\Delta_n \leq \frac{1}{|2\pi i|} \int_\gamma \frac{|f(z)||h|^{n+1}|dz|}{|z-a|^{n+1}|z-a-h|}$$

$$\leq \frac{M}{2\pi} \int_\gamma \frac{|h|^{n+1}|dz|}{\delta_2{}^{n+1}(\delta_2-\delta_1)}$$

$$= \frac{M|h|^{n+1}}{2\pi\delta_2{}^{n+1}(\delta_2-\delta_1)} 2\pi\delta_2 = \frac{M|h|}{(\delta_2-\delta_1)}\left(\frac{|h|}{\delta_2}\right)^n$$

Since $|h| \leq \delta_1 < \delta_2$, it follows that as $n \to \infty, \Delta_n \to 0$

So that we have the identity

$$f(a+h) = f(a) + \sum_{n=1}^{\infty} \frac{f^n(a)}{\lfloor n} h^n$$

Changing over from a + h to z, thus we have the so called Taylor's series (expansion)

$$f(z) = f(a) + \sum_{n=1}^{\infty} \frac{f^n(a)}{\lfloor n} (z-a)^n$$

So far, we have proved only the convergence of this series for all values of z such that $|z-a| \leq \delta_1$,

It is however possible to prove more i.e. the uniform convergence as follows. Since $|h| \leq \delta_1$, we have

$$|\Delta_n| \leq \frac{M\delta_1}{\delta_1-\delta_2}\left(\frac{\delta_1}{\delta_2}\right)^n$$

and we observe that the expression on the right is independent of h. Therefore, given $\in >0$, there exists an integer $N = N(\in)$, independent of h, such that $|\Delta_n| <\in$ for $n \geq N$. This proves the uniform convergence of the Taylor`s series of $f(z)$ in the region $|z - a| \leq \delta_1 < \delta$

2.1.36 Remarks:

(i) The above theorem is sometimes known as the **Cauchy-Taylor theorem.**

(ii) By putting a = 0, Taylor's expansion reduces to

$$f(z) = f(0) + \sum_{n=1}^{\infty} \frac{f^n(0)}{\lfloor n} z^n$$, which is known as **Maclaurin's series**.

(iii) Taylor`s series can be put as

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

where

$$a_n = \frac{f^n(z)}{\lfloor n} = \frac{1}{\lfloor n}.\frac{\lfloor n}{2\pi i}\int \frac{f(z)}{(z-a)^{n+1}}dz$$

$$= \frac{1}{2\pi i}\int \frac{f(z)}{(z-a)^{n+1}}dz.$$

(iv) Using $a_n = \dfrac{f^n(a)}{\lfloor n}$, the result of **Cauchy's inequality** (2.1.31) can be put as

$$|a_n| = \left|\frac{f^n(a)}{\lfloor n}\right| \le \frac{M\lfloor n}{\lfloor n R^n} = \frac{M}{R^n}$$

i.e. $\quad |a_n| \le \dfrac{M}{R^n}$.

2.1.37Theorem: On the circumference of the circle of convergence of a power series, there must be at least one singular point of the function represented by the series.

Proof: Suppose that there is no singularity on the circumference |z – a| = R of the radius of convergence of the power series.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

Then, the function $f(z)$ will be regular in a disc $|z-a| < R + \in$, where $\in$ is sufficiently small positive number. But from this it follows that the series $\sum_{n=0}^{\infty} a_n (z-a)^n$ must converge in the disc $|z-a| < R + \in$ and this contradicts the assumption that $|z-a| < R$ is the circle of convergence.

Hence, there is at least one singular point of the function $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ on the circle of convergence of the power series $\sum_{n=0}^{\infty} a_n (z-a)^n$.

2.1.38 Example: Expand the following functions in a Taylor's series about point z = 0 and determine the region of convergence

(i) e^z (ii) $\sin z$ (iii) $\cos z$

Solution: (i) Let $f(z) = e^z$,

$$f'(z) = e^z, \ldots\ldots\ldots, f^{(r)}(z) = e^z$$

$$\therefore f(0) = 1, f'(0) = 1, \ldots., f^{(r)}(0) = 1$$

By Taylor's series, we have

$$f(z)=e^z=f(0)+zf'(0)+\frac{z^2}{2!}f''(0)+....+\frac{z^r}{r!}f^{(r)}(0)+.......$$

$$=1+z+\frac{z^2}{2!}+....+\frac{z^r}{r!}+.......$$

Here, $u_r=\frac{z^r}{r!}, u_{r+1}=\frac{z^{r+1}}{(r+1)!}$

$$\lim_{r\to\infty}\left|\frac{u_r}{u_{r+1}}\right|=\lim_{r\to\infty}\frac{1}{|z|}(r+1)=\infty.$$

By ratio test, the series is convergent.

(ii) Let $f(z)=\sin z$

$$f'(z)=\cos z=\sin(\frac{\pi}{2}+z)$$

$$f''(z)=-\sin z=\sin(\pi+z)$$

………………………

………………………

$$f^{(r)}(z)=\sin(\frac{r\pi}{2}+z)$$

$\therefore f(0)=0,\ f'(0)=1,\ f''(0)=0........$

By Taylor's series, we have

$$f(z)=\sin z=f(0)+zf'(0)+\frac{z^2}{2!}f''(0)+....+\frac{z^r}{r!}f^{(r)}(0)+.......$$

$$=0+z+\frac{z^2}{2!}(0)+\frac{z^3}{3!}(-1)....+\frac{z^r}{r!}(-1)^r+.......=z-\frac{z^3}{3!}+\frac{z^5}{5!}+......$$

Here, $|u_r|=\frac{z^{2r-1}}{(2r-1)!}, |u_{r+1}|=\frac{z^{2r+1}}{(2r+1)!}$

$$\lim_{r\to\infty}\left|\frac{u_r}{u_{r+1}}\right|=\lim_{r\to\infty}\frac{1}{z^2}(2r+1)(2r)=\infty.$$

So, series is convergent everywhere.

(iii) Let $f(z)=\cos z$

$$f'(z) = -\sin z = \cos(\frac{\pi}{2} + z)$$

$$f''(z) = -\cos z = \cos(\pi + z)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$f^{(r)}(z) = \cos(\frac{r\pi}{2} + z)$$

By Taylor's series, we have

$$f(z) = \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \dots\dots$$

Here, $\left|u_r\right| = \frac{z^{2r-2}}{(2r-2)!}, \left|u_{r+1}\right| = \frac{z^{2r}}{(2r)!}$

$$\lim_{r\to\infty}\left|\frac{u_r}{u_{r+1}}\right| = \lim_{r\to\infty}\frac{1}{z^2}(2r-1)(2r) = \infty > 1$$

So, series is convergent everywhere.

2.1.39 Example: Expand log (1 + z) in a Taylor's series about the point z = 0 and determine the region of convergence for the resulting series.

Solution: Let $f(z) = \log(1+z)$.Then

$$f(z) = \frac{1}{1+z}, f'(z) = -\frac{1}{(1+z)^2}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$f^n(z) = \frac{(-1)^{n-1}\underline{\lfloor n-1}}{(1+z)^n}$$

Hence, $f(0) = 0, f'(0) = 1, f''(0) = -1$

$$f^n(0) = (-1)^{n-1}\underline{\lfloor n-1}$$

Therefore, by Taylor's theorem,

$$f(z) = \log(1+z) = f(0) + zf'(0) + \frac{z^2}{\underline{\lfloor 2}}f''(0) + \dots + \frac{z^n}{\underline{\lfloor n}}f^n(0) + \dots$$

$$= 0 + z + \frac{z^2}{\underline{\lfloor 2}}(-1) + \dots + \frac{z^n}{\underline{\lfloor n}}(-1)^{n-1}\underline{\lfloor n-1} + \dots\dots$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} ..... + (-1)^{n-1} \frac{z^n}{n} + ...$$

Now, if u_n denotes the n^{th} term of the series, then

$$u_n = \frac{(-1)^{n-1} z^n}{n}, u_{n+1} = \frac{(-1)^n z^{n+1}}{n+1}$$

$$\therefore \qquad \lim_{x \to \infty} \left| \frac{u_n}{u_{n+1}} \right| = \frac{1}{|z|}$$

Hence, by ratio test, the series converges for $\frac{1}{|z|} > 1$ i.e. $|z| < 1$.

2.1.40 Example: If the function *f*(z) is analytic when |z| < Rand has the Taylor's expansion $\sum_{n-0}^{\infty} a_n z^n$, show that if r<R,

$$\frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$$

Hence, prove that if $|f(z)| \le M$ when $|z| < R$, $\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \le M^2$.

Solution: Since *f*(z) is analytic for |*z*| < *R*, so *f*(z) is analytic within and on a closed contour C defined by |*z*| = *r, r* < *R*. Thus *f*(z) can be expanded in a Taylor's series within |*z*| = *r* so that

$$f(z) = \sum_0^{\infty} a_n z^n$$

$$= \sum_0^{\infty} a_n r^n e^{in\theta}, z = re^{i\theta}$$

$$\therefore |f(z)|^2 = f(z)\overline{f(z)} \sum_{n=0}^{\infty} a_n r^n e^{in\theta} \sum_{m=0}^{\infty} \overline{a_m} r^m e^{-im\theta}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n \overline{a_m} r^{m+n} e^{-i(n-m)\theta}$$

The two series for *f*(z) and $\overline{f(z)}$ are absolutely convergent and hence their product is uniformly convergent for the range 0 ≤ 0 ≤ 2π. Thus, the term by term integration is justified. So, we get

$$\int_0^{2\pi} |f(z)|^2 d\theta = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n \overline{a_m} r^{m+n} \int_0^{2\pi} e^{i(n-m)\theta} d\theta$$

$$= \sum_{n=0}^{\infty} a_n \overline{a_n} r^{n+n} .2\pi, \int_0^{\infty} e^{i(n-m)\theta} d\theta = \begin{cases} 0, & n \neq m \\ 2\pi, & n = m \end{cases}$$

or $$\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^2 d\theta = \sum_{n+0}^{\infty} |a_n|^2 r^{2n} \quad (1)$$

Now, from (1), we get

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^2 d\theta$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} M^2 d\theta = \frac{1}{2\pi} M^2 2\pi = M^2 .$$

Which proves the required result.

2.1.41Example: If a function *f*(z) is analytic for all finite values of z and as |z|→∞, |*f*(z)| = A|z|^K, then *f*(z) is a polynomial of degree ≤K.

Solution: Here, *f*(z) is analytic in the finite part of z-plane. Also, it is given that

$$\lim_{|z|\to\infty} |f(z)| = A|z|^K \quad (1)$$

We can assume that f(z) is analytic inside a circle C defined by |z|=R, where R is large but finite. Hence f(z) can be expanded in a Taylor's series as

$$f(z) = \sum_0^{\infty} a_n z^n \text{, where } a_n = \frac{f(0)}{\lfloor n} = \frac{\lfloor n}{\lfloor n\, 2\pi i}\int_C \frac{f(z)}{(z-0)^{n+1}} dz = \frac{1}{2\pi i}\int_C \frac{f(z)}{z^{n+1}} dz$$

Therefore, $$|a_n| \leq \frac{1}{2\pi i}\int_C \frac{|f(z)|\,|dz|}{|z|^{n+1}} = \frac{1}{2\pi R^{n+1}}\int_C |f(z)||dz|$$

$$\leq \frac{M}{2\pi R^{n+1}}\int_C |dz|, M = \max .|f(z)| \, on \, C.$$

$$= \frac{M}{2\pi R^{n+1}} 2\pi R = \frac{M}{R^n} = \frac{A|z|^K}{R^n}$$

$$= \frac{AR^K}{R^n} = \frac{A}{R^{n-K}} .$$

Thus, $|a_n| \leq AR^{K-n} = \dfrac{A}{R^{n-K}}$

which tends to zero as $R \to \infty$, if $n - K > 0$ i.e. $a_n = 0$ for all n such that $n > K$. Now, from (2), we conclude that *f(z)* is a polynomial of degree ≤ K. Hence the result.

SECTION-III

3.1 Laurent's Series: Consider the functions which are analytic in a punctured disc i.e. an open disc with centre removed. We have seen that a function $f(z)$ which is regular in an open disc at z = a, can be expanded in a Taylor's series in powers of ($z - a$) and that this power series is convergent in any circular region with centre 'a', contained within the given neighbourhood.

In case, however, the function is not analytic in the neighbourhood of a point 'a' including it, but analytic only in a ring shaped region (sometimes called annulus) surrounding 'a', the expansion of $f(z)$ in a Taylor's series in powers of ($z - a$) ceases to be valid. The question naturally arises as to whether $f(z)$, for values of z in the above said ring shaped region, can be expanded in powers of ($z - a$) or not. In such a situation $f(z)$ has another series expansion known as Laurents's expansion.

3.1.1 Definition: Circles lying in the same plane and having a common centre is called **concentric circles** and the region between two concentric circles is called an **annulus**.

3.1.2 Definition (Weierstrass's M-test): Suppose $\sum u_n(z)$ is an infinite series of single valued functions defined in a bounded closed domain D. Let there exist a series $\sum M_n$ of positive constants independent of z such that

(i) $|u_n(z)| \le M_n \quad \forall\ n$ and $\forall\ z \in D$

(ii) $\sum M_n$ is convergent

Then, the series $\sum u_n(z)$ is uniformly and absolutely convergent in the domain *D.*

3.1.3 Laurent's Theorem: Let $f(z)$ be analytic in the ring shaped region between two concentric circles C and C' of radii R and R' $(R' < R)$ with centre 'a', and on the circles themselves, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n},$$

z being any point of the annulus and

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-a)^{n+1}}\,dw,$$

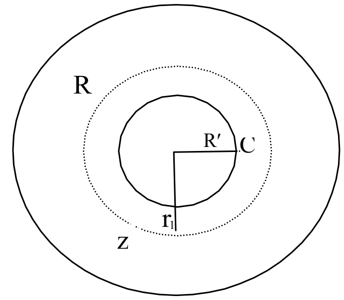
$$b_n = \frac{1}{2\pi i}\int_{C'} \frac{f(w)}{(w-a)^{-n+1}}\,dw.$$

Proof: Since $f(z)$ is analytic on the circles and within the annulus between two circles. So by Cauchy's integral formula, we have

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\int_{C'} \frac{f(w)}{w-z}\,dw \ . \qquad (1)$$

Consider the identity

$$\frac{1}{w-z}=\frac{1}{(w-a)-(z-a)}=\frac{1}{(w-a)\left[1-\dfrac{z-a}{w-a}\right]} \tag{2}$$



Applying the result,

$$\frac{1}{1-b}=1+b+b^2+\ldots+b^{n-1}+\frac{b^n}{1-b}=\sum_{r=0}^{n-1}b^r+\frac{b^n}{1-b}$$

on R.H.S. of (2), we obtain

$$\frac{1}{w-z}=\frac{1}{w-a}\left[\sum_{r=0}^{n-1}\left(\frac{z-a}{w-a}\right)^r+\left(\frac{z-a}{w-a}\right)^n\frac{1}{1-\dfrac{z-a}{w-a}}\right]$$

$$=\sum_{r=0}^{n-1}\frac{(z-a)^r}{(w-a)^{r+1}}+\left(\frac{z-a}{w-a}\right)^n\frac{1}{w-z} \tag{3}$$

Interchanging z and w, we get

$$\frac{1}{z-w}=\sum_{r=0}^{n-1}\frac{(w-a)^r}{(z-a)^{r+1}}+\frac{(w-a)^n}{(z-a)^n}\frac{1}{z-w} \tag{4}$$

Equations (3) and (4) can be written as

$$\frac{f(w)}{w-z}=\sum_{r=0}^{n-1}\frac{(z-a)^r}{(w-a)^{r+1}}f(w)+\left(\frac{z-a}{w-a}\right)^n\frac{f(w)}{w-z}\qquad \forall\, w \text{ on } C \tag{5}$$

$$\frac{f(w)}{z-w}=\frac{-f(w)}{w-z}=\sum_{r=0}^{n-1}\frac{(w-a)^r}{(z-a)^{r+1}}f(w)+\left(\frac{w-a}{z-a}\right)^n\frac{f(w)}{z-w}\qquad \forall\, w \text{ on } C' \tag{6}$$

Let M and M' be the maximum values of $|f(w)|$ on C and C' respectively. Also let $|z-a|=r_1$. Equations of circles C and C' are $|w-a|=R$ and $|w-a|=R'$ respectively.

From the figure, it is clear that

$$\left.\begin{aligned}\left|\frac{w-a}{z-a}\right|&=\frac{R'}{r_1}<1 \quad \text{if } w \text{ lies on } C'\\ \left|\frac{z-a}{w-a}\right|&=\frac{r_1}{R}<1 \quad \text{if } w \text{ lies on } C\end{aligned}\right\} \tag{7}$$

The absolute value $|u_n(z)|$ of general term of the series in (5) is

$$|u_n(z)|=\left|\frac{(z-a)^n}{(w-a)^{n+1}}f(w)\right|$$

$$\le\frac{r_1^n}{R^{n+1}}M=\frac{M}{R}\left(\frac{r_1}{R}\right)^n$$

Similarly, the absolute value $\left|u_n'(z)\right|$ of general term of the series (6) is

$$\left|u_n'(z)\right| \le \frac{(R')^n}{r_1^{n+1}} M' = \frac{M'}{r_1}\left(\frac{R'}{r_1}\right)^n$$

Hence, the series of positive terms

$$\Sigma \frac{M}{R}\left(\frac{r_1}{R}\right)^n \ \textit{and}\ \Sigma \frac{M'}{r_1}\left(\frac{R'}{r_1}\right)^n \quad \text{are both convergent as } \frac{r_1}{R} < 1,\ \frac{R'}{r_1} < 1.$$

Consequently by Weierstrass M-test, both the series in (5) and (6) are uniformly convergent. Hence term by term integration is valid. Integrating (5) and (6), we obtain

$$\frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}dw = \sum_{r=0}^{n-1} \frac{(z-a)^r}{2\pi i}\int_C \frac{f(w)}{(w-a)^{r+1}}dw + \frac{(z-a)^n}{2\pi i}\int_C \frac{f(w)}{(w-a)^n(w-z)}dw$$

and

$$-\frac{1}{2\pi i}\int_{C'} \frac{f(w)}{w-z}dw = \sum_{r=0}^{n-1} \frac{(z-a)^{-r-1}}{2\pi i}\int_{C'} f(w)(w-a)^r dw + \frac{1}{(z-a)^n 2\pi i}\int_{C'} \frac{(w-a)^n}{z-w} f(w)dw.$$

Taking

$$a_r = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-a)^{r+1}}$$

$$b_{r+1} = \frac{1}{2\pi i}\int_C (w-a)^r f(w)$$

and

$$U_n = \frac{1}{2\pi i}\int_C \left(\frac{z-a}{w-a}\right)^n \frac{f(w)}{w-z}dw,$$

$$V_n = \frac{1}{2\pi i}\int_{C'} \left(\frac{w-a}{z-a}\right)^n \frac{f(w)}{z-w}dw.$$

We get

$$\frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}dw = \sum_{r=0}^{n-1} (z-a)^r a_r + U_n \tag{8}$$

$$-\frac{1}{2\pi i}\int_{C'} \frac{f(w)}{w-z}dw = \sum_{r=0}^{n-1} \frac{b_{r+1}}{(z-a)^{r+1}} + V_n \tag{9}$$

Adding (8) and (9) and using (1), we get

$$f(z) = \sum_{r=0}^{n-1} a_r(z-a)^r + \sum_{r=1}^{n} b_r(z-a)^{-r} + U_n + V_n \tag{10}$$

Now,

$$|U_n| = \left| \frac{1}{2\pi i} \int_C \left(\frac{z-a}{w-a} \right)^n \frac{f(w)}{w-z} dw \right|$$

$$\leq \frac{1}{2\pi} \int_C \left(\frac{r_1}{R} \right)^n \frac{M|dw|}{R-r_1} \qquad \left(|w-z| = |(w-a)-(z-a)| \geq |w-a| - |z-a| = R - r_1 \right)$$

$$= \frac{1}{2\pi} \left(\frac{r_1}{R} \right)^n \frac{M}{R-r_1} 2\pi R$$

$$= \frac{M}{1-\frac{r_1}{R}} \left(\frac{r_1}{R} \right)^n ,$$

which tends to zero as $n \to \infty$, since $\frac{r_1}{R} < 1$. Thus, $\lim_{n\to\infty} U_n = 0$. Similarly, we can get $\lim_{n\to\infty} V_n = 0$.

Making $n \to \infty$ in (10), we obtain

$$f(z) = \sum_{r=0}^{\infty} a_r (z-a)^r + \sum_{r=1}^{\infty} b_r (z-a)^{-r}$$

or

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n} \qquad (11)$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-a)^{n+1}} dw ,$$

$$b_n = \frac{1}{2\pi i} \int_{C'} \frac{f(w)}{(w-a)^{-n+1}} dw$$

which proves the theorem.

3.1.4 Remarks:

(i) The result (11) can be put in a more compact form as $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$, where the co-efficients are given by the single formula $a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{n+1}} dw$ where γ denotes C when $n \geq 0$ and C' when $n < 0$ since however the integrand is analytic in the annulus $R' < |z-a| < R$, we may take γ to be any closed contour which passes round the ring.

(ii) The function $f(z)$ which is expanded in Laurent's series is one-valued. Laurent's theorem will not provide an expansion for multi-valued function.

(iii) In the particular case, when $f(z)$ is analytic inside C', all the coefficients b_n are zero, by Cauchy's theorem, and the series reduces to Taylor's series.

(iv) The series of positive powers of $z - a$ converges, not merely in the ring, but everywhere inside the circle C. Similarly, the series of negative powers of $z - a$ converges everywhere outside C'.

(v) The series of negative powers of $z-a$ i.e., $\sum_{n=1}^{\infty} b_n (z-a)^{-n}$ is called the principal part of Laurent's expansion, while the series of positive powers i.e. $\sum_{n=0}^{\infty} a_n (z-a)^n$ is called the regular part.

(vi) There is no handy method, like that for Taylor's series, for finding the Laurent coefficients. But if we can find them by any method (generally by direct expansion), their validity is justified due to the fact that Laurent's co-efficients are unique.

3.1.5 Example: Expand $f(z)=\dfrac{1}{(z+1)(z+3)}$ in a Laurent's series valid for the regions.

(i) $|z|<1$ (ii) $1<|z|<3$ (iii) $|z|>3$ (iv) $0<|z+1|<2$

Solution: Resolving into partial fractions, we get

$$f(z)=\frac{1}{(z+1)(z+3)}=\frac{1}{2(z+1)}-\frac{1}{2(z+3)}$$

(i) For $|z|<1$, we have

$$f(z)=\frac{1}{2}(z+1)^{-1}-\frac{1}{2}(z+3)^{-1}$$

$$=\frac{1}{2}(z+1)^{-1}-\frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$=\frac{1}{2}[1-z+z^2-z^3+\ldots]-\frac{1}{6}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3+\ldots\right]$$

$$=\frac{1}{3}-\frac{4}{9}z+\frac{13}{27}z^2$$

(ii) For $|z|>1$, we have

$$\frac{1}{2(z+1)}=\frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1}=\frac{1}{2z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\ldots\right]$$

$$=\frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}-\frac{1}{2z^4}+\ldots$$

and for $|z|<3$, *we have*

$$\frac{1}{2(z+3)}=\frac{1}{6\left(1+\frac{z}{3}\right)}=\frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$=\frac{1}{6}-\frac{z}{18}+\frac{z^2}{54}-\frac{z^3}{162}+\ldots$$

Hence, the Laurent's series for *f (z)*, valid for the annulus $1<|z|<3$, is

$$f(z)=\ \ldots-\frac{1}{2z^4}+\frac{1}{2z^3}-\frac{1}{2z^2}+\frac{1}{2z}-\frac{1}{6}+\frac{z}{18}-\frac{z^2}{54}+\frac{z^3}{162}+\ldots$$

(iii) For $|z| > 3$, we have

$$f(z) = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1} = \frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4}...$$

(iv) Take z + 1 = u, then $0 < |u| < 2$ and we have

$$f(z) = \frac{1}{(z+1)(z+3)} = \frac{1}{u(u+2)}$$

$$= \frac{1}{2u\left(1+\frac{u}{2}\right)} = \frac{1}{2u}\left(1+\frac{u}{2}\right)^{-1}$$

$$= \frac{1}{2u} - \frac{1}{4} + \frac{u}{8} - \frac{u^2}{16} + ...$$

$$= \frac{1}{2(z+1)} - \frac{1}{4} + \frac{z+1}{8} - \frac{(z+1)^2}{16} + ...$$

3.1.6 Example: By considering the Laurent's series for the function $f(z) = \dfrac{1}{(1-z)(z-2)}$. Prove that

(i) If C is any closed contour within annulus $1 < |z| < 2$, then $\int_C f(z)\,dz = 2\pi i$.

(ii) If C is any closed contour which contains both the points $z = 1$ and $z = 2$ in its interior, then

$\int_C f(z)\,dz = 0$.

Solution: Resolving $f(z)$ into partial fractions, we get

$$f(z) = \frac{1}{(1-z)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

(i) In the annulus $1 < |z| < 2$, $\dfrac{1}{|z|} < 1$ and $\dfrac{|z|}{2} < 1$, so that

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z\left(1-\frac{1}{z}\right)} + \frac{1}{2\left(1-\frac{z}{2}\right)}$$

$$= \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+...\right) + \frac{1}{2}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+...\right) = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$$

Now, in the region $1<|z|<2$, the function $f(z)$ is analytic and has the Laurent's expansion

$$f(z)=\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}+\sum_{n=1}^{\infty}\frac{1}{z^n} \qquad (1)$$

If we write this as $f(z)=\sum_{n=-\infty}^{\infty}a_n z^n$, then $a_n=\dfrac{1}{2\pi i}\int_C f(z)z^{-(n+1)}dz$, $n\in 0,\pm1,\pm2........$ (2)

where C is any closed contour within the annulus $1<|z|<2$. Since, in the expansion (1), the coefficient of z^{-1} is 1, so we get from (2),

$\int_C f(z)\,dz=2\pi i$. [By equating coefficients of z^{-1} in (1) and (2)].

(ii) In the domain $|z|>2$, $\dfrac{1}{|z|}<1$ and $\dfrac{2}{|z|}<1$, so that Laurent's series is

$$f(z)=\frac{1}{(1-z)(z-2)}=\frac{1}{z-1}-\frac{1}{z-2}=\frac{1}{z\left(1-\frac{1}{z}\right)}-\frac{1}{z\left(1-\frac{2}{z}\right)}$$

$$=\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}-\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$=\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+........\right)-\frac{1}{z}\left(1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+........\right)$$

$$=\sum_{n=0}^{\infty}\frac{1-2^n}{z^{n+1}}\ ,\quad |z|>2 \qquad (3)$$

Since the coefficient of z^{-1} in this expansion is 0, therefore from (2) and (3), we get

$\int_\gamma f(z)\,dz=0$,

where γ is any closed contour in the region $|z|>2$. Since $f(z)$ is analytic except at points $z=1$ and $z=2$, it follows by well-known result on complex integration that

$\int_c f(z)\,\mathrm{dz}=\int_\gamma f(z)\,dz=0$,

where C is any simple closed that contains both the points $z=1$ and $z=2$ in its interior.

3.1.7 Example: For the function $f(z)=\dfrac{2z^3+1}{z^2+z}$, find the Taylor's series valid in the neighbourhood of the point $z=i$ and a Laurent's series valid within the annulus with centre at the origin.

Solution. Here, $f(z)=\dfrac{2z^3+1}{z^2+z}=2(z-1)+\dfrac{1}{z+1}+\dfrac{1}{z}$ (1)

To find the Taylor's expansion in the neighbourhood of point $z=i$ for the function $f(z)$, let us take

$$f(z)=g(z)+\phi(z)+\gamma(z), \qquad (2)$$

where $g(z)=2(z-1),\ \phi(z)=\dfrac{1}{z+1},\ \gamma(z)=\dfrac{1}{z}$.

(i)Taylor's expansion for $g(z)$ is

$$g(z)=\sum_{n=0}^{\infty} a_n(z-i)^n, \text{ where } a_n=\frac{g^{(n)}(i)}{n!} .$$

Here,

$$\begin{array}{ll} g(z)=2(z-1), & \therefore\ g(i)=2(i-1) \\ g'(z)=2, & g'(i)=2 \\ g^{(n)}(z)=0\quad \forall n\geq 2, & g^{(n)}(i)=0\quad \forall n\geq 2 \end{array}$$

$\therefore a_0=2(i-1),\ a_1=2,\ a_n=0\ \ \forall\ n\geq 2$

$$\therefore\quad g(z)=2(i-1)+2(z-i) \tag{3}$$

(ii) Taylor's expansion for $\gamma(z)=\dfrac{1}{z}$ is

$$\gamma(z)=\sum_{n=0}^{\infty} a_n(z-i)^n, \text{ where } a_n=\frac{\gamma^{(n)}(i)}{n!} .$$

Here

$$\begin{array}{ll} \gamma(z)=\dfrac{1}{z}, & \therefore\ \gamma(i)=\dfrac{1}{i} \\ \gamma'(z)=-\dfrac{1}{z^2}, & \gamma'(i)=-\dfrac{1}{(i)^2} \\ \gamma^{(n)}(z)=\dfrac{(-1)^n\, n!}{z^{n+1}}\quad , & \gamma^{(n)}(i)=\dfrac{(-1)^n\, n!}{(i)^{n+1}} \end{array}$$

Therefore, $a_n=\dfrac{(-1)^n}{(i)^{n+1}}$

$$\text{Hence, } \gamma(z)=\sum_{n=0}^{\infty}\frac{(-1)^n(z-i)^n}{(i)^{n+1}} \tag{4}$$

(iii) Taylor's expansion for $\phi(z)$ is

$$\phi(z)=\sum_{n=0}^{\infty} a_n(z-i)^n, \text{ where } a_n=\frac{\phi^{(n)}(i)}{n!} .$$

Here,

$$\phi(z)=\frac{1}{z+1},\qquad \therefore\ \phi(i)=\frac{1}{i+1}$$

$$\phi'(z) = -\frac{1}{(z+1)^2}, \qquad \phi'(i) = -\frac{1}{(i+1)^2}$$

$$\phi^{(n)}(z) = \frac{(-1)^n \, n!}{(z+1)^{n+1}}, \qquad \phi^{(n)}(i) = \frac{(-1)^n \, n!}{(i+1)^{n+1}}$$

$$\therefore \; a_n = \frac{(-1)^n}{(i+1)^{n+1}}$$

$$\therefore \; \phi(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z-i)^n}{(i+1)^{n+1}} \qquad (5)$$

Now, writing equation (2) with the help of (3), (4) and (5), we obtain the required Taylor's expansion of $f(z)$ as

$$f(z) = 2(i-1) + 2(z-i) + \sum_{n=0}^{\infty} (-1)^n (z-i)^n \left(\frac{1}{i^{n+1}} + \frac{1}{(i+1)^{n+1}} \right)$$

Laurent's expansion for $f(z)$ in the annulus $0 < |z| < 1$ is

$$f(z) = 2(z-1) + \frac{1}{z} + (1+z)^{-1}$$

$$= 2(z-1) + \frac{1}{z} + \sum_{n=0}^{\infty} (-1)^n z^n .$$

3.1.8 Example: Show that $e^{\frac{c}{2}\left(z-\frac{1}{z}\right)} = \sum_{n=-\infty}^{\infty} a_n z^n$ *where* $a_n = \frac{1}{2\pi} \int_0^{2\pi} \cos(n\theta - c\sin\theta) d\theta$.

Solution: The function $f(z) = e^{\frac{c}{2}\left(z-\frac{1}{z}\right)}$ is analytic except at z = 0 and z = ∞ . Hence $f(z)$ is analytic in the annulus $r_1 \le |z| \le r_2$, where r_1 is small and r_2 is large. Therefore, $f(z)$ can be expanded in the Laurent's series in the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n} \qquad (1)$$

where $a_n = \frac{1}{2\pi i} \int_{C'} \frac{f(z)}{z^{n+1}} dz$,

$b_n = \frac{1}{2\pi i} \int_{C'} \frac{f(z)}{z^{-n+1}} dz$,

C' being any circle with centre at the origin for the sake of convenience, let us take C' to be the unit circle |z| = 1 which gives $z = e^{i\theta}$. Now,

$$a_n = \frac{1}{2\pi i}\int_{C'} \frac{e^{\frac{c}{2}(z-z^{-1})}}{z^{n+1}}dz,$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{ci\sin\theta} ie^{i\theta}}{e^{i(n+1)\theta}}d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{i(c\sin\theta - n\theta)}d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos(c\sin\theta - n\theta)d\theta + \frac{i}{2\pi}\int_0^{2\pi} F(\theta)d\theta\ , \qquad (2)$$

where $F(\theta) = \sin\left(c\sin\theta - n\theta\right)$ .Since

$$F(2\pi-\theta) = \sin[c\sin(2\pi-\theta) - n(2\pi-\theta)]$$
$$= -\sin(c\sin\theta - n\theta + 2n\pi)$$
$$= -\sin(c\sin\theta - n\theta) = -F(\theta)$$

$$\therefore \quad \int_0^{2\pi} F(\theta)d\theta = 0$$

Thus, from (2), we have

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos(c\sin\theta - n\theta)d\theta = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - c\sin\theta)d\theta\,.$$

We note that if z is replaced by z^{-1}, the function $f(z)$ remains unaltered so that $b_n = (-1)^n a_n$

Hence, from (1), we get

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty}(-1)^n a_n z^{-n}$$
$$= \sum_{n=-\infty}^{\infty} a_n z^n .$$

Where, $a_n = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \cos(n\theta - c\sin\theta)d\theta$.

3.1.9 Example: Prove that the function $f(z)$= cosh(z + z $^{-1}$) can be expanded in a series of the type $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$ in which the co-efficients of z^n and z^{-n} , both are given by

$$\frac{1}{2\pi}\int_0^{2\pi} \cos n\theta \cosh(2\cos\theta)d\theta.$$

Solution: The function $f(z)$= cosh (z + z $^{-1}$) is analytic except at z = 0 and z =∞ . Hence $f(z)$ is analytic in the annulus $r_1 \le |z| \le r_2$, where r_1 is small and r_2 is large. Therefore, $f(z)$ can be expanded in the Laurent's seriesas

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n} \qquad (1)$$

Where

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{z^{n+1}}dz,$$

$$b_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{z^{-n+1}}dz,$$

where C being any circle with centre at the origin. Let us take C to be the unit circle $|z| = 1$ which gives $z = e^{i\theta}$. Now,

$$a_n = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\cosh(z+\overline{z})}{z^{n+1}}dz$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{\cosh(2\cos\theta)e^{i\theta}}{e^{i(n+1)\theta}} id\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)e^{-in\theta}d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\cos n\theta\, d\theta - \frac{i}{2\pi}\int_0^{2\pi} F(\theta)d\theta \qquad (2)$$

where $F(\theta) = \cosh(2\cos\theta)\sin n\theta$

We note that

$$F(2\pi-\theta) = -F(\theta) \;\Rightarrow\; \int_0^{2\pi} F(\theta)d\theta = 0$$

Thus, (2) becomes

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\cos n\theta\; d\theta\;. \qquad (3)$$

It is clear that

$$b_n = a_{-n} = \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\cos(-n\theta)\; d\theta = a_n\,.$$

Thus, from (1), we find

$\cosh(z+z^{-1}) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_n z^{-n} = a_0 + \sum_{n=1}^{\infty} a_n (z^n + z^{-n})$, where a_n is given by (3).

3.1.10 Example: Prove that the function $f(z) = \sin\left[c\,\left(z + z^{-1}\right)\right]$ can be expanded in a series of the type $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$ in which the co-efficients of z^n and z^{-n}, both are given by

$$\frac{1}{2\pi}\int_0^{2\pi} \sin(2c\cos\theta)\cos n\theta\, d\theta.$$

Solution: The function is analytic except at points $z = 0$ and $z = \infty$. Hence, $f(z)$ is analytic in the annulus $r_1 \le |z| \le r_2$, where r_1 is small and r_2 is large.

Therefore, $f(z)$ can be expanded in the Laurent's seriesas

$$f(z)=\sin[c(z+z^{-1})]=\sum_{n=0}^{\infty}a_n z^n+\sum_{n=1}^{\infty}b_n z^{-n} \tag{1}$$

where

$$a_n=\frac{1}{2\pi i}\int_C \frac{f(z)}{z^{n+1}}dz=\frac{1}{2\pi i}\int_C \frac{\sin[c(z+z^{-1})]}{z^{n+1}}dz,$$

$$b_n=\frac{1}{2\pi i}\int_C \frac{f(z)}{z^{-n+1}}dz=\frac{1}{2\pi i}\int_C \frac{\sin[c(z+z^{-1})]}{z^{-n+1}}dz,$$

and C being any circle with centre at the origin. We take C to be the unit circle $|z|=1$ which gives $z=e^{i\theta} \Rightarrow dz=ie^{i\theta}d\theta$.

Therefore,

$$a_n=\frac{1}{2\pi i}\int_0^{2\pi}\frac{\sin[c(e^{i\theta}+e^{-i\theta})]}{e^{i(n+1)\theta}}\,ie^{i\theta}\,d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\frac{\sin[c(e^{i\theta}+e^{-i\theta})]}{e^{in\theta}.e^{i\theta}}\,e^{i\theta}\,d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\sin[c(2\cos\theta)]e^{-in\theta}d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\sin[c(2\cos\theta)]\cos n\theta\,d\theta-\frac{i}{2\pi}\int_0^{2\pi}F(\theta)d\theta \tag{2}$$

where $F(\theta)=\sin[c(2\cos\theta)]\sin n\theta$.

We note that

$$F(2\pi-\theta)=-F(\theta)$$

$$\Rightarrow \int_0^{2\pi}F(\theta)d\theta=0$$

Thus, (2) becomes

$$a_n=\frac{1}{2\pi}\int_0^{2\pi}\sin[c(2\cos\theta)]\cos n\theta\ d\theta. \tag{3}$$

It is clear that

$b_n=a_{-n}=\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\sin[c(2\cos\theta)]\cos(-n\theta)\ d\theta=a_n$. Thus, from (1), we find

$$\sin[c(z+z^{-1})]=\sum_{n=0}^{\infty}a_n z^n+\sum_{n=1}^{\infty}b_n z^{-n}\ ,$$

where $a_n=b_n=\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\sin(2c\cos\theta)\cos(n\theta)\ d\theta$.

3.1.11 Example: Find the Laurent series for $f(z)=\dfrac{z}{z^2+1}$ around $z_0=i$. Give the region where your answer is valid.

Solution: Using partial fractions, we have

$$f(z)=\frac{z}{z^2+1}=\frac{z}{(z+i)(z-i)}=\frac{1}{2}\left[\frac{1}{z-i}+\frac{1}{z+i}\right].$$

Since $\frac{1}{z+i}$ is analytic at z = i, it has a Taylor series expansion. We find it using geometric series.

$$\frac{1}{z+i}=\frac{1}{2i}\frac{1}{1+\frac{z-i}{2i}}=\frac{1}{2i}\sum_{n=0}^{\infty}\left(-\frac{z-i}{2i}\right)^n$$

So, the Laurent series is

$$f(z)=\frac{1}{2}\cdot\frac{1}{z-i}+\frac{1}{4i}\sum_{n=0}^{\infty}\left(-\frac{z-i}{2i}\right)^n$$

The region of convergence is $0<|z-i|<2.$

3.1.12 Exercise:

1. Prove that $\frac{1+2z}{z^2+z^3}=\frac{1}{z^2}+\frac{1}{z}-1+z-z^2+z^3...,\qquad 0<|z|<1.$
2. Expand $\frac{1}{z(z^2-3z+2)}$ for the regions

   (i) $0<|z|<1$ (ii) $1<|z|<2$ (iii) $|z|>2$
3. Expand $\frac{(z-2)(z+2)}{(z+1)(z+4)}$ for the regions

   (i) $|z|<1$ (ii) $1<|z|<4$ (iii) $|z|>4$
4. Expand $\frac{z^2-1}{(z+2)(z+3)}$ for the regions

   (i) $|z|<2$ (ii) $2<|z|<3$ (iii) $|z|>3$
5. Expand $\frac{1}{z(1+z^2)}$ for the regions

   (i) $0<|z|<1$ (ii) $|z|>1$
6. Find the Laurent's series of the function $f(z)=\frac{1}{z^2(1-z)}$ about z = 0.
7. Expand $\frac{1}{(1+z^2)(z+2)}$ for the regions

   (i) $|z|<1$ (ii) $1<|z|<2$ (iii) $|z|>2$

8. Expand $\dfrac{z+3}{z(z^2-z-2)}$ for the regions

(i) $|z|<1$ (ii) $1<|z|<2$ (iii) $|z|>2$

3.2 Zeros of an analytic function: A zero of an analytic function $f(z)$ is the value of z such that $f(z)=0$. Suppose $f(z)$ is analytic in a domain D and a is any point in D. Then, by Taylor's theorem, $f(z)$ can be expanded about $z = a$ in the form

$$f(z)=\sum_{n=0}^{\infty} a_n (z-a)^n \text{ , where } a_n = \frac{f^n(a)}{n!} \tag{1}$$

Suppose $a_0 = a_1 = a_2 = \ldots\ldots = a_{m-1} = 0, \;\; a_m \neq 0$ (2)

so that $f(a) = f'(a) = \ldots\ldots = f^{m-1}(a) = 0, \;\; f^m(a) \neq 0$

In this case, we say that $f(z)$ has a zero of order m at $z = a$ and thus (1) takes the form

$$\begin{aligned} f(z) &= \sum_{n=m}^{\infty} a_n (z-a)^n \\ &= \sum_{n=0}^{\infty} a_{n+m} (z-a)^{n+m} \\ &= (z-a)^m \sum_{n=0}^{\infty} a_{n+m} (z-a)^n \end{aligned}$$

Let $\sum_{n=0}^{\infty} a_{n+m}(z-a)^n = \phi(z)$. (3)

Therefore, we get $f(z)=(z-a)^m \phi(z)$ (4)

Now,

$$\begin{aligned} \phi(a) &= \left[\sum_{n=0}^{\infty} a_{n+m} (z-a)^n \right]_{z=a} \\ &= \left[a_m + \sum_{n=1}^{\infty} a_{n+m} (z-a)^n \right]_{z=a} = a_m \ . \end{aligned}$$

Since $a_m \neq 0$, *so* $\phi(a) \neq 0$. Thus, an analytic function $f(z)$ is said to have a zero of order m at $z = a$ if $f(z)$ is expressible as $f(z)=(z-a)^m \phi(z)$, where $\phi(z)$ is analytic and $\phi(a) \neq 0$. Also, $f(z)$ is said to have a simple zero at $z = a$ if $z = a$ is a zero of order one.

3.2.1 Theorem: Zeros are isolated points.

Proof: Let us take the analytic function $f(z)$ which has a zero of order m at $z = a$. Then, by definition, $f(z)$ can be expressed as

$$f(z)=(z-a)^m \phi(z) \text{ , where } \phi(z) \text{ is analytic and } \phi(a) \neq 0 .$$

Let $\phi(\mathrm{a}) = 2K$. Since $\phi(z)$ is analytic in sufficiently small neighbourhood of a, it follows from the continuity of $\phi(z)$ in this neighbourhood that we can choose δ so small that, for $|z-a|<\delta$,

$|\phi(z)-\phi(a)|<|K|$.

Hence,

$$\begin{aligned}|\phi(z)| &= |\phi(a)+\phi(z)-\phi(a)| \\ &\geq |\phi(a)|-|\phi(z)-\phi(a)| \\ &> |2K|-|K| \\ &= |K|, \text{ for } |z-a|<\delta\end{aligned}$$

Since $K \neq 0$ thus, $\phi(z)$ does not vanish in the region $|z-a|<\delta$. Since $f(z)=(z-a)^m\phi(z)$, it follows at once that $f(z)$ has no zero other than 'a' in the same region. Thus, we conclude that there exists a neighbourhood of 'a' in which the only zero of $f(z)$ is the point 'a' itself i.e. 'a' is an isolated zero.

The above theorem can also be stated as, "Let $f(z)$ be analytic in a domain *D*, then unless $f(z)$ is identically zero, there exists a neighbourhood of each point in *D* throughout which the function has no zero except possibly at the point itself."

3.3 Isolated Singularity: The point where the function ceases to be analytic is called the singularity of the function. Suppose that a function $f(z)$ is analytic throughout the neighbourhood of a point z=a, say for $|z-a|<\delta$, except at the point *a* itself. Then the point '*a*' is called an isolated singularity of the function $f(z)$. In other words, the point $z = a$ is said to be isolated singularity of $f(z)$ if $f(z)$ is not analytic at $z = a$ and there exists a deleted neighbourhood of $z = a$ containing no other singularity.

For example, the function $f(z)=\dfrac{z+1}{(z-1)(z+2)(z+3)}$ has three isolated singularities at z = 1,-2,-3 respectively and the function $f(z)=\dfrac{1}{z}$ is analytic everywhere except at origin. Therefore, z = 0 is an isolated singularity.

3.3.1 Definition: Let $z = a$ be an isolated singularity of the function $f(z)$, then there exists a deleted neighbourhood of point $z=a$ in which $f(z)$ is analytic. Thus, if z is any point in this neighbourhood, then by Laurent's expansion $f(z)$ can be written as

$$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\sum_{n=1}^{\infty}b_n(z-a)^{-n},$$

where $\sum_{n=1}^{\infty}b_n(z-a)^{-n}$ is the principal part of the expansion of $f(z)$ at the singular point $z=a$.

There are now three possible cases, discussed as follows.

3.3.2 Removable Singularity: If the principal part of $f(z)$ at z = a contains no term i.e. $b_n = 0$, for all n, then the singularity $z = a$ is called a removable singularity of $f(z)$. In such a case, the singularity can be removed by defining the function at $z = a$ in such a way that it becomes analytic there. For example, the function $f(z)=\dfrac{\sin z}{z}$ is undefined at z = 0. Also, we have

$$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{\lfloor 3} + \frac{z^5}{\lfloor 5}\ldots\ldots\ldots \right)$$

$$= 1 - \frac{z^2}{\lfloor 3} + \frac{z^4}{\lfloor 5}\ldots\ldots\ldots\ldots$$

Thus, $\frac{\sin z}{z}$ contains no negative powers of *z*. If it were the case $f(0) \neq 1$, then *z* = 0 is a removable singularity which can be removed by simply redefining $f(0) = 1$. This function is purely analytic.

3.3.3 Pole: If the principal part of $f(z)$ at $z = a$ contains a finite number of terms, say *m*, i.e. $b_n = 0$ for all *n* such that *n* > *m*, then the singularity is called a pole of order *m*. Poles of order 1, 2, 3 are called simple, double, triple poles. The coefficient b_1 is called the residue of $f(z)$ at the pole *a*.

Thus, if $z=a$ is a pole of order *m* of the function $f(z)$, then $f(z)$ has the expansion of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{m} b_n (z-a)^{-n}$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \ldots + \frac{b_m}{(z-a)^m}$$

$$= \frac{1}{(z-a)^m}\left[b_m + b_{m-1}(z-a) + \ldots + b_1 (z-a)^{m-1} + \sum_{n=0}^{\infty} a_n (z-a)^{m+n} \right]$$

$$= \frac{\phi(z)}{(z-a)^m},$$

where $\phi(z)$ is analytic for $|z-a| < R$ and $\phi(a) = b_m \neq 0$. Hence, if $f(z)$ has a pole of order *m* at *z* = *a*, then $|f(z)| \to \infty$ as $z \to a$ in any manner, i.e., an analytic function cannot be bounded in the neighbourhood of a pole.

For example, the function $f(z) = \frac{z-2}{(z-5)^2 (z+4)^3}$ has z = 5, z = -4 as poles of order two and three respectively.

3.3.4 Isolated Essential Singularity: If the principal part of $f(z)$ at $z = a$ has an infinite number of terms, i.e., $b_n \neq 0$ for infinitely many values of *n*, then the singularity *a* is called isolated essential singularity or essential singularity. In this case, *a* is evidently also a singularity of $\frac{1}{f(z)}$.

For example, $e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{\lfloor 2 z^2} + \frac{1}{\lfloor 3 z^3} + \ldots$ has z = 0 as an isolated essential singularity.

3.3.5 Example: What kind of singularity has the function $f(z) = \frac{e^z}{z^2+4}$?

Solution: Poles of $f(z)$ are given by $z^2 + 4 = 0 \quad \Rightarrow \quad z = \pm 2i$. Hence, z = 2i and z = -2i are simple poles i.e., poles of order 1.

3.3.6 Theorem: If an analytic function $f(z)$ has a pole of order m at $z=a$, then $\frac{1}{f(z)}$ has a zero of order m at $z=a$ and conversely.

Solution: Suppose that the analytic function $f(z)$ has a pole of order m at $z=a$. We shall prove that $\frac{1}{f(z)}$ has a zero of order m at $z=a$. By definition, $f(z)$ is expressible as

$$f(z)=\frac{\phi(z)}{(z-a)^m}$$

$$\text{i.e., } (z-a)^m f(z)=\phi(z), \tag{1}$$

where $\phi(z)$ is analytic and $\phi(a)\neq 0$.

From (1), we get $$\frac{1}{f(z)}=\frac{(z-a)^m}{\phi(z)} \tag{2}$$

Making $z\to a$ in (2) and noting that $\phi(a)\neq 0$, we get $\frac{1}{f(z)}=0$ as $z\to a$. This implies that $\frac{1}{f(z)}$ has a zero of order *m*. Conversely, suppose that $\frac{1}{f(z)}$ has a zero of order *m*, we shall prove that $f(z)$ has a pole of order *m*. By definition, $\frac{1}{f(z)}$ is expressible as

$$\frac{1}{f(z)}=(z-a)^m\psi(z) \quad , \tag{3}$$

where $\psi(z)$ is analytic and $\psi(a)\neq 0$.

From (3), we get $(z-a)^m f(z)=\frac{1}{\psi(z)}$. Taking $\frac{1}{\psi(z)}=\phi_1(z)$

$$(z-a)^m f(z)=\phi_1(z) \tag{4}$$

Since $\psi(z)$ is analytic, therefore $\phi_1(z)$ is also analytic and $\psi(a)\neq 0 \Rightarrow \phi_1(a)\neq 0$. The equation (4) proves that $f(z)$ has a pole of order *m*, since $f(z)=\frac{\phi_1(z)}{(z-a)^m}$.

Further, note that poles are isolated, since zeros are isolated.

3.3.7 Example: Show that e^{-1/z^2} has no singularity.

Solution: Here, $f(z)=e^{-1/z^2}=\frac{1}{e^{1/z^2}}$. Poles of $f(z)$ are given by $e^{1/z^2}=0$, which is not possible for any value of z real or complex.

Now, zeros of $f(z)$ are given by $e^{-1/z^2}=0=e^{-\infty}$ so that $\frac{1}{z^2}=\infty \;\Rightarrow\; z=0$.

Here, z is a zero of order two so that there is no singularity. Thus, we conclude that $f(z)$ is free from any singularity.

3.3.8 Example: If a function $f(z)$ is analytic for all values of z and if $|f(z)| \geq \delta > 0$ for some constant δ which is strictly positive, then prove that $f(z)$ is constant.

Solution: Consider the function $g(z) = \dfrac{1}{f(z)}$. Since $|f(z)| \geq \delta$, therefore

$$|g(z)| = \frac{1}{|f(z)|} \leq \frac{1}{\delta} \leq M \ ,$$

where M is any positive number. Hence, by Liouville's theorem which states that, " A function regular in all finite regions of complex plane and bounded is equal to a constant". We have $g(z)$ and hence $f(z)$ will be constant provided that $g(z)$ is analytic for all values of z. This is so in view of the fact that $f(z)$ is analytic for all values of z. Hence the result.

3.3.9 Example: Find the singularities of the function $f(z) = \dfrac{e^{\frac{c}{z-a}}}{e^{\frac{z}{a}} - 1}$, indicating the character of each singularity.

Solution: Here

$$f(z) = \frac{e^{\frac{c}{z-a}}}{e^{\frac{z}{a}} - 1} = \frac{\exp\left(\dfrac{c}{z-a}\right)}{\exp\left(1 + \dfrac{z-a}{a}\right) - 1}$$

$$= \frac{e^{\frac{c}{z-a}}}{e.e^{\frac{z-a}{a}} - 1} = -e^{\frac{c}{z-a}}\left[1 - e.e^{\frac{z-a}{a}}\right]^{-1}$$

$$= -e^{\frac{c}{z-a}}\left[1 - e\left\{1 + \frac{z-a}{a} + \frac{(z-a)^2}{\lfloor 2\, a^2} + \ldots\right\}\right]^{-1}$$

$$= -\left[1 + \frac{c}{z-a} + \left(\frac{c}{z-a}\right)^2 \frac{1}{\lfloor 2} + \ldots\right] \times \left[1 + e\left\{1 + \frac{z-a}{a} + \ldots\right\} + e^2\left\{1 + \frac{z-a}{a} + \ldots\right\}^2 + ..\right]$$

Clearly, this expansion contains positive and negative powers of $(z - a)$. Moreover, terms containing negative powers of $(z - a)$ are infinite in number. Hence, by definition, $z = a$ is an isolated essential singularity.

Again, $f(z) = \dfrac{e^{\frac{c}{z-a}}}{e^{\frac{z}{a}} - 1}$

Evidently, denominator has zero of order one at $e^{\frac{z}{a}} = 1 = e^{2n\pi i}$ i.e. $z = 2n\pi ia$. Thus, $f(z)$ has a pole of order one at each point $z = 2n\pi ia$, where n = 0, $\pm$ 1, $\pm$ 2…

3.3.10 Example: Find zeros and poles of

$$f(z) = \left(\frac{z+1}{z^2+1}\right)^2$$

Solution: Zeros of $f(z)$ are given by $f(z) = 0$ i.e.,

$$f(z) = \left(\frac{z+1}{z^2+1}\right)^2 = 0 \quad \Rightarrow \quad (z+1)^2 = 0 \quad \Rightarrow \; z = -1, -1.$$

Hence, z = -1 is a zero of order 2.

Poles of $f(z)$ are given by putting the denominator of $f(z)$ zero i.e.,

$$(z^2+1)^2 = 0 \quad \Rightarrow \; (z+i)^2(z-i)^2 = 0 \qquad \Rightarrow \; z = -i, -i, i, i.$$

Thus, z = i and z = -i are poles each of order 2.

3.3.11 Example: Discuss the nature of singularities of the function $f(z) = \tan z$.

Solution: We have $f(z) = \tan z = \dfrac{\sin z}{\cos z}$. Hence, to obtain the singularities of $f(z)$, the denominator of $f(z)$ equating to zero, we get

$$\cos z = 0 \quad \Rightarrow \quad z = 2n\pi \pm \frac{\pi}{2}, \quad n \in \mathbb{Z}$$

$$\Rightarrow z = (4n \pm 1)\frac{\pi}{2}, n \in \mathbb{Z} \quad \Rightarrow z = (2n \pm 1)\frac{\pi}{2}, \; n \in \mathbb{Z}$$

Hence, $z = (2n \pm 1)\dfrac{\pi}{2}$, $n \in \mathbb{Z}$ are the simple poles of f(z).

3.3.12 Example: A function which has no singularity in finite part of the complex plane or at ∞ is constant.

Solution: Suppose that the function $f(z)$

(i) has no singularity in the finite part of the *z*-plane.

(ii) has no singularity at $z = \infty$.

Due to (i), $f(z)$ can be expanded in a Taylor's series about $z = 0$ in the form $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$, where z is any point inside or on $|z| = k$, where k is an arbitrary positive real number, thus

$$f\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n z^{-n} \qquad (1)$$

From (ii), we note that $f\left(\frac{1}{z}\right)$ has no singularity at $z=0$, so $f\left(\frac{1}{z}\right)$ can be expanded in a Taylor's series as

$$f\left(\frac{1}{z}\right)=\sum_{n=0}^{\infty} b_n z^n \qquad (2)$$

From (1) and (2), we get

$$\sum_{n=0}^{\infty} a_n z^{-n}=\sum_{n=0}^{\infty} b_n z^n$$

This is possible only if (i) $a_n = 0 = b_n \quad \forall\, n \geq 1$

(ii) $a_0 = b_0$

This implies that $f\left(\frac{1}{z}\right)=a_0=b_0=$ constant and thus $f(z)$ is constant.

3.3.13 Example: Show that a function which has no singularities in the finite part of the complex plane and has a pole of order *n* at $z=\infty$, is a polynomial of degree *n*.

Solution: Suppose that the function $f(z)$

(i) has no singularity in the finite part of the complex plane,

(ii) has a pole of order *n* at $z=\infty$.

Due to (i), $f(z)$ can be expanded in a Taylor's series about $z=0$ in the form of

$$f(z)=\sum_{m=0}^{\infty} a_m z^m$$

$$\Rightarrow \quad f\left(\frac{1}{z}\right)=\sum_{m=0}^{\infty} a_m z^{-m} \qquad (1)$$

From (ii), we note that $f\left(\frac{1}{z}\right)$ has a pole of order *n* at $z=0$ i.e., principal part of Laurent's expansion of $f\left(\frac{1}{z}\right)$ contains only *n* terms. Thus,

$$f\left(\frac{1}{z}\right)=\sum_{m=0}^{\infty} b_m z^m+\sum_{m=1}^{n} c_m z^{-m} \qquad (2)$$

From (1) and (2), we get

$$\sum_{m=0}^{\infty} a_m z^{-m}=\sum_{m=0}^{\infty} b_m z^m+\sum_{m=1}^{n} c_m z^{-m}$$

$$\Rightarrow \quad a_0=\sum_{m=0}^{\infty} b_m z^m \text{ and } a_m=c_m \;\; \forall\, m \text{ such that } 1 \leq m \leq n \text{ and } a_{n+\gamma}=0 \text{ for } \gamma=1,2,\ldots$$

Thus, (1) becomes

$$f\left(\frac{1}{z}\right)=\sum_{m=0}^{n} a_m z^{-m} \Rightarrow \quad f(z)=\sum_{m=0}^{n} a_m z^{m}.$$

This implies that $f(z)$ is a polynomial of degree n.

3.3.14Theorem (Limiting Points of Zeros): Let $f(z)$ be a function regular in a domain D . Let $\alpha_1, \alpha_2, ..., \alpha_n, ...$ be a set of points having a limiting point α in the interior of D. If $f(z)=0$ at points $\alpha_1, \alpha_2, ..., \alpha_n, ...$, it follows that either $f(z)$ vanishes identically throughout the interior of D or $f(z)$ has an isolated essential singularity at $z=\alpha$.

Proof: Since $f(z)$ is analytic so that it is continuous function having zeros at $\alpha_1, \alpha_2, ..., \alpha_n, ...$ Thus, every neighbourhood of the point $z=\alpha$ containing zeros of the function. Therefore, $z=\alpha$ must be zero of $f(z)$.

But we know that zeros are isolated, therefore α cannot be a zero of $f(z)$. Hence, $f(z)$ must be identically zero. Now, considering the case $f(z) \neq 0$, then $z=\alpha$ is a singularity of $f(z)$. Hence, in this case the singularity is isolated but it is not a pole, since $f(z)$ does not tend to ∞ as z tends to α in any manner, so that the limit point of zeros must be an isolated essential singularity of $f(z)$.

3.3.15 Remarks: The following two results are direct consequences of the above theorem.

(i) If a function is regular in a region and vanishes at all points of a sub region of the given region, or along any arc of a continuous curve in the region, then it must be identically zero throughout the interior of the given region.

(ii) If two functions are regular in a region, and have identical values at an infinite number of points which have a limiting point in the region, they must be equal to each other throughout the interior of the given region i.e. If two functions, which are analytic in a domain, coincide in a part of that domain, then they coincide in the whole domain. For this, we take $f(z)=f_1(z)-f_2(z)$.

3.3.16 Limiting Point of Poles: Let $f(z)$ be analytic except at a set of points which are poles say $\alpha_1, \alpha_2, ..., \alpha_n, ...$ having a limit point 'α'in the region D. Thus, every neighbourhood of the point $z=\alpha$ containing pole of $f(z)$. Thus, $z=\alpha$ cannot be a pole and it is not isolated (Since poles are isolated). Such a singularity is called non-isolated essential singularity or simply essential singularity.

3.3.17 Example: Find the kind of singularities of the following functions

(i) $\dfrac{\cot \pi z}{(z-a)^2}$ at $z=a$ and $z=\infty$.

(ii) $\tan\left(\dfrac{1}{z}\right)$ at $z=0$

(iii) $\sin\left(\dfrac{1}{1-z}\right)$ at $z=1$

Solution: (i) Here, $f(\text{z})=\dfrac{\cot \pi z}{(\text{z}-\text{a})^2}=\dfrac{\cos \pi z}{\sin \pi z\ (\text{z}-\text{a})^2}$.

Poles of $f(\text{z})$ are given by $(\text{z}-\text{a})^2 \sin \pi z=0$.

$\Rightarrow \quad (\text{z}-\text{a})^2=0$ or $\sin \pi z=0$

$\Rightarrow \quad \text{z}=a, a \quad$ or $\quad \pi z=n\pi,\ \ n\in 0,\pm 1,\pm 2........$

$\Rightarrow \quad \text{z}=a, a \quad$ or $z=n,\ \ n\in 0,\pm 1,\pm 2......$

Obviously, $z=\infty$ is the limit point of these poles. Hence, $z=\infty$ is a non-isolated essential singularity as limit point of a pole is a non-isolated essential singularity. Also, $z=a$ is a double pole of order 2.

(ii) Here, $f(\text{z})=\tan\left(\dfrac{1}{z}\right)=\dfrac{\sin\left(\dfrac{1}{z}\right)}{\cos\left(\dfrac{1}{z}\right)}$. Poles of $f(\text{z})$ are given by $\cos\left(\dfrac{1}{z}\right)=0$

$$\Rightarrow \qquad \frac{1}{z}=2n\pi\pm\frac{\pi}{2},\ \ n\in 0,\pm 1,\pm 2,....$$

$$\Rightarrow \qquad z=\frac{1}{2n\pi+\dfrac{\pi}{2}},\ \ n\in 0,\pm 1,\pm 2,.....$$

Clearly, z = 0 is the limit point of these poles. Hence, z = 0 is a non-isolated singularity.

(iii) Here, $f(\text{z})=\sin\left(\dfrac{1}{1-z}\right)$, zeros of $f(z)$ are given by $\sin\left(\dfrac{1}{1-z}\right)=0$

$$\Rightarrow \quad \frac{1}{1-z}=n\pi,\ \ n\in 0,\pm 1,\pm 2,.....$$

$$\Rightarrow \quad z=1-\frac{1}{n\pi},\ \ n\in 0,\pm 1,\pm 2,.....$$

Clearly, $z=1$ is the limit point of these zeros. Hence, $z=1$ is an isolated essential singularity

3.3.18 Exercise: Find the poles of the following functions:

(i) $\dfrac{1}{(z^2+1)^3(z-1)^4}$ (ii) $z\cot z$ (iii) $\dfrac{\sin z}{z^5}$ (iv) $\dfrac{1}{1-e^z}$ (v) $\dfrac{z}{1-e^z}$

3.3.19 Behaviour of an Analytic Function near an Isolated Essential Singularity: As we know that if $z=a$ is a pole of an analytic function $f(z)$, then $|f(z)|\to\infty$ as $z\to a$ in any manner. The behaviour of an analytic function near an isolated essential singularity is of much complicated character. The following theorem is a precise statement of this complicated nature of *f (z)* near an isolated essential singularity and this theorem is called Weierstrass theorem.

3.3.20 Weierstrass Theorem: If 'a' is an isolated essential singularity of $f(z)$, then given positive numbers l, ε, however small, and any number K, however large, there exists a point z in the circle $|\text{z} - \text{a}| < l$ at which $|f(z) - \text{K}| < \varepsilon$

(or)

In any neighbourhood of an isolated essential singularity, an analytic function approaches any given value arbitrarily closely.

Proof: We first observe that if l and M are any positive numbers, then there are values of z in the circle $|z - a| < l$ at which

$$|f(z)| > M \qquad (1)$$

For, if this were not true, then we would have $|f(z)| \leq M$ for $|z-a| < l$. If the principal part in the Laurent expansion of $f(z)$ about a is $\sum_{n=1}^{\infty} b_n (z-a)^{-n}$, where $b_n = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^{-n+1}} dw$ and γ is the circle $|w - a| = r$, r being sufficiently small, then

$$\begin{aligned} |b_n| &= \left| \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^{-n+1}} dw \right| \\ &= \left| \frac{1}{2\pi i}\int_\gamma (w-a)^{n-1} f(w)\, dw \right| \\ &\leq \frac{M}{2\pi} r^{n-1} \int_\gamma |dw| \\ &= \frac{M}{2\pi} r^{n-1}\, 2\pi r = M r^n . \end{aligned}$$

By the result of the absolute value of a complex integral, this holds for all $n \geq 1$ and r, so that, making $r \to 0$, we find that $b_n = 0$ for $n \geq 1$. This implies that there is no isolated essential singularity at $z = a$. But this contradicts the hypothesis that a is an isolated essential singularity of $f(z)$. Thus, the observed result (1) is true, i.e., "in the neighbourhood of an isolated essential singularity, $f(z)$ cannot be bounded."

Now, let us take any finite, but arbitrary positive number K. There are now two distinct possibilities, either $f(z)$ - K has zeros inside every circle $|z - a| = l$ or else we can find a sufficiently small l such that $f(z)$ - K has no zero for $|z - a| < l$. In the first case, the result follows immediately. In the second case, choosing a sufficiently small l, we have

$$|f(z) - K| \neq 0 \text{ in } |z - a| < l, \text{ so that}$$

$$\phi(z) = \frac{1}{f(z) - K}$$

is regular for $|z - a| < l$, except at a whereas we shall just see, $\phi(z)$ has an essential singularity.

We have $f(z) = \frac{1}{\phi(z)} + K$.

If $\phi(z)$ were analytic at a, $f(z)$ would either be analytic or have a pole at a. On the other hand, if $\phi(z)$ has a pole at a, $f(z)$ would be obviously analytic there. Thus, we reach at the contradiction and therefore, $\phi(z)$ has an essential singularity at a. So, due to (1), given $\varepsilon > 0$, there exists a point z in the circle |z - a | <l such that $|\phi(z)| > \frac{1}{\in}$ *i.e.*, $|f(z)-K| < \in$ .Hence, theorem is proved.

3.3.21 Remark: The theorem 3.3.19 helps us to understand clearly the distinction between poles and isolated essential singularities. While $|f(z)| \to \infty$, as z tends to a pole in any manner, at an isolated essential singularity $f(z)$ has no unique limiting value, and it comes arbitrarily close to any arbitrarily pre assigned value at infinity of points in every neighbourhood of the isolated essential singularity.

3.4 Maximum Modulus Principle: Here, we continue the study of properties of analytic functions. Contrary to the case of real functions, we cannot speak of maxima and minima of a complex function $f(z)$, since ℂis not an ordered field. However, it is meaningful to consider maximum and minimum values of the modulus $|f(z)|$of the complex function $f(z)$, real part of $f(z)$and imaginary part of $f(z)$. The following theorem known as maximum modulus principle, is also true if $f(z)$ is not one-valued, provided $|f(z)|$ is one-valued.

3.4.1 Maximum Modulus Theorem: Let $f(z)$ be analytic within and on a simple closed contour *C*. If $|f(z)| \leq M$ on *C*, then the inequality $|f(z)| < M$ holds everywhere within *C*. Moreover $|f(z)|$ = M at a point within *C* if and only if $f(z)$ is constant.

In other words, $|f(z)|$ attains the maximum value on the boundary *C* and not at any interior point of the region *D* bounded by *C*.

Proof: We prove the theorem by contradiction. If possible, let $|f(z)|$ attain the maximum value at an interior point $z = z_0$ of the region *D* enclosed by *C*. Since $f(z)$ is analytic inside *C*, we can expand $f(z)$ by Taylor's theorem in the neighbourhood of pointz_0 as

$$f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{where } a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^n}\,dz$$

and γ is the circle $|z - z_0| = r$, r being small.

We have $|z-z_0| = re^{i\theta}$ i.e., $z = z_0 + re^{i\theta}$, $0 \leq \theta \leq 2\pi$.

Also,

$$\begin{aligned} |f(z)|^2 &= f(z)\overline{f(z)} \\ &= \sum_{n=0}^{\infty} a_n r^n e^{in\theta} \sum_{m=0}^{\infty} \overline{a}_m r^m e^{-im\theta} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n \overline{a}_m r^{n+m} e^{i(n-m)\theta} \end{aligned}$$

Integrating both sides from 0 to 2π , we get

$$\begin{aligned} \int_0^{2\pi} |f(z)|^2\, d\theta &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n \overline{a}_m r^{m+n} \int_0^{2\pi} e^{i(n-m)\theta} \\ &= \sum_{n=0}^{\infty} a_n \overline{a}_n r^{2n}\, 2\pi, \qquad n = m \\ &= \sum_{n=0}^{\infty} |a_n|^2 r^{2n}\, 2\pi, \qquad (1) \end{aligned}$$

where $\int_0^{2\pi} e^{i(n-m)\theta} d\theta = \begin{cases} 0 & if\ \ n \neq m \\ 2\pi & if\ \ n = m \end{cases}$

From (1), we have for n = 0,

$$\int_0^{2\pi} |f(z)|^2\, d\theta = |a_0|^2\, 2\pi$$

and putting $z = z_0$in this, we find

$$\int_0^{2\pi} |f(z_0)|^2\, d\theta = |a_o|^2\, 2\pi$$

$$\Rightarrow\ |f(z_0)|^2 \int_0^{2\pi} d\theta = |a_0|^2\, 2\pi$$

$$\Rightarrow |f(z_0)|^2\ 2\pi = |a_0|^2\, 2\pi\ \ \Rightarrow |f(z_0)|^2 = |a_0|^2 \tag{2}$$

Also, since $f(z)$ has maximum value at z = z_0, so

$$|f(z)|^2 \leq |f(z_0)|^2 = |a_0|^2 .$$

Hence from (1), we get

$$\sum_{n=0}^{\infty} |a_n|^2\, r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} |f(z)|^2\, d\theta$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)|^2\, d\theta$$

$$= \frac{1}{2\pi} |a_0|^2\, 2\pi = |a_0|^2$$

Thus, $|a_0|^2 + |a_1|^2 r^2 + |a_2|^2 r^4 + ... \leq |a_0|^2$ for positive values of r.

Hence, $|a_1| = |a_2| = |a_3| = ... = 0$

$$i.e.,\ a_1 = a_2 = a_3 = ... = 0 .$$

Which implies $f(z)$ = a_0 = constant.

Hence, $|f(z)|$ cannot attain a maximum value at an interior point of D which is a contradiction to our supposition. Also $|f(z)|$ attains a maximum value at an interior point of D if it is constant and in that case, $|f(z)| = M$ throughout D.

3.4.2 Minimum Modulus Principle: Let $f(z)$ be analytic within and on a simple closed contour C and let $f(z) \neq 0$ inside C. Further suppose that $f(z)$ is not constant, then $|f(z)|$ cannot attain a minimum value inside C.

Proof: Since $f(z)$ is analytic within and on C and also $f(z) \neq 0$ inside C, so $\frac{1}{f(z)}$ is also analytic within and on C. Therefore, by maximum modulus principle, $\left|\frac{1}{f(z)}\right|$ cannot attain a maximum value inside C which implies that $|f(z)|$ cannot have a minimum value inside C.

3.4.3 Theorem: Let $f(z)$ be an analytic function, regular for |z| < R and let $M(r)$ denote the maximum of $|f(z)|$ on $|z| = r$, then $M(r)$ is a steadily increasing function of r for $r < R$.

Proof: By maximum modulus principle, for two circles

$|z| = r_1$ and $|z| = r_2$, we have

$|f(z)| \leq M(r)$, where $r_1 < r_2$

which implies $M(r_1) \leq M(r_2)$, $r_1 < r_2$

and $M(r_1) = M(r_2)$ if $f(z)$ is constant.

Also $M(r)$ cannot be bounded because if it were so, then $f(z)$ is a constant (by Liouville's theorem). Hence $M(r)$ is a steadily increasing function of r.

3.4.4 Schwarz's Lemma: Let $f(z)$ be analytic in a domain D defined by |z| < R and let $|f(z)| \leq M$ for all z in D and $f(0) = 0$, then $|f(z)| \leq \frac{M}{R}|z|$.

Also, if the equality holds for any one z, then $f(z) = \frac{M}{R} z e^{i\alpha}$ where α is real constant.

Proof: Let C be the circle |z| = r <R.

Since $f(z)$ is analytic within and on C, therefore by Taylor's theorem

$f(z) = \sum_{n=0}^{\infty} a_n z^n$ at any point z within C.

i.e., $f(z) = a_0 + a_1 z + a_2 z^2 + \ldots$

Under the assumption $f(0) = 0$, we get $a_0 = 0$

$\therefore \; f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \ldots \qquad (1)$

Let $g(z) = \dfrac{f(z)}{z} \qquad (2)$

then we have

$$g(z) = a_1 + a_2 z + a_3 z^2 + \ldots \qquad (3)$$

The function $g(z)$ in (2) has a singularity at $z = 0$ which can be removed if we define $g(z) = a_1$ for $z = 0$ i.e. $g(0) = a_1$. Now, $g(z)$ is analytic within and on C and so by maximum modulus principle, $|g(z)|$ attains maximum value on C, say at $z = z_0$ and not within C.

Thus, $|z_0| = r < R$ and

$$|g(z_0)| = \left|\frac{f(z_0)}{z_0}\right| = \max.\, value\, of\, |g(z)| = \left|\frac{f(z)}{z}\right| \leq \frac{M}{r} \qquad (4)$$

and thus for any z inside C, we have

$$|g(z)| < \frac{M}{r}$$

i.e., $$\left|\frac{f(z)}{z}\right| < \frac{M}{r} \Rightarrow |f(z)| < \frac{M}{r}|z| \qquad (5)$$

This inequality holds for all *r* such that r< *R*.

Now, L. H. S. is free from *r*, making $r \to R$ in (5), we find

$$|f(z)| < \frac{M}{R}|z| \quad \forall\ z \text{ such that } |z| < R\ .$$

Also, from (4), we note that for the point z_0 on *C*,

$$|f(z_0)| = \frac{M}{r}|z_0|$$

Making $r \to R$, we get

$$|f(z_0)| = \frac{M}{R}|z_0|$$

i.e., $f(z) = \frac{M}{R} z e^{i\alpha}$ *for z lying on* $|z| = R.$

Which proves the result.

3.4.5 Remarks:

(i) If we take $M = 1$, $R = 1$, then Schwarz's lemma takes the form as follows, "If $f(z)$ is analytic in a domain *D* defined by $|z|<1$ and $|f(z)| \le 1$ for all *z* in *D* and $f(0) = 0$, then $|f(z)| \le |z|$.

Also if the equality holds for any one *z*, then $f(z) = ze^{i\alpha}$, where α is a real constant."

(ii) In view of the power series expansion,

$$f(z) = f(0) + z f'(0) + \frac{z^2}{\lfloor 2} f''(0) + \ldots$$

We get

$$\frac{f(z)}{z} = f'(0) + \frac{z^2}{\lfloor 2} f''(0) + \ldots$$

where we have assumed that $f(z)$ satisfies the conditions of Schwarz's lemma so that

$$\left|\frac{f(z)}{z}\right| \le \frac{M}{R}$$

This implies that

$$\left|f'(0) + \frac{z^2}{\lfloor 2} f''(0) + \ldots\ldots\right| \le \frac{M}{R}$$

By setting $z = 0$, we obtain $f'(0) \le \frac{M}{R}$

(iii) Let $f(z)$ be analytic inside and on the unit circle, $|f(z)| \le M$ on the circle and $f(0) = a$ where $0 < a < m$. Then

$$|f(z)| \le M \frac{M|z|+a}{a|z|+M}$$

inside the circle.

For its proof, we consider

$$\phi(z) = M \frac{f(z)-a}{a\,f(z)-M^2}$$

Then, $\phi(0) = M \dfrac{f(0)-a}{a\,f(0)-M^2} = M \dfrac{a-a}{a^2-M^2} = 0$

Also, $\phi(z)$ is regular at every point on the unit circle.

Also, $|\phi(z)| = \left| M \dfrac{f(z)-a}{a\,f(z)-M^2} \right| \le \left| M \dfrac{M-a}{aM-M^2} \right| = 1$.

Thus, $\phi(z)$ satisfies all the conditions of Schwarz's lemma.

Therefore,

$$|\phi(z)| = \left| M \frac{f(z)-a}{a\,f(z)-M^2} \right| \le |z|$$

which gives

$$|f(z)| \le M \frac{M|z|+a}{a|z|+M}.$$

3.5 Meromorphic Function: A function $f(z)$ is said to be meromorphic in a region D if it is analytic in D except at a finite number of poles. In other words, a function $f(z)$ whose only singularities in the entire complex plane are poles, is called a meromorphic function. The word meromorphic is used for the phrase "analytic except for poles". The concept of meromorphic is used in contrast to holomorphic.A meromorphic function is a ratio of entire functions. Rational functions are meromorphic functions. e.g

$$\begin{aligned} f(z) &= \frac{z^2-1}{z^5+2z^3+z} = \frac{(z+1)(z-1)}{z(z^4+2z^2+1)} \\ &= \frac{(z+1)(z-1)}{z(z^2+1)^2} = \frac{(z+1)(z-1)}{z(z+i)^2(z-i)^2} \end{aligned}$$

has poles at $z = 0$ (simple), at $z = \pm i$ (both double) and zeros at $z = \pm 1$ (both simple). Since only singularities of $f(z)$ are poles, therefore $f(z)$ is a meromorphic function.

Similarly, *tan z*, *cot z*, *sec z* are all meromorphic functions.

3.5.1 Remark: A meromorphic function does not have essential singularity.

3.5.2 Theorem: Let $f(z)$ be analytic inside and on a simple closed contour *C* except for a finite number of poles inside *C* and let $f(z) \neq 0$ on *C*, then $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N - P$.

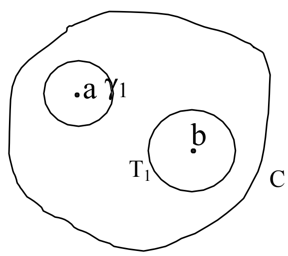
Where *N* and *P* are respectively the total number of zeros and number of poles of $f(z)$ inside *C*, a zero (pole) of order *m* being counted *m* times.

Proof: Suppose that $f(z)$ is analytic within and on a simple closed contour *C* except at a pole $z = a$ of order *p* inside *C* and also suppose that $f(z)$ has a zero of order *n* at $z = b$ inside *C*.

Then, we have to prove that

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = n - p$$

Let γ_1 and T_1 be the circles inside *C* with centres at $z = a$ and $z = b$ respectively.



Then, by cor. to Cauchy's theorem, we have

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f'(z)}{f(z)}dz + \frac{1}{2\pi i}\int_{T_1} \frac{f'(z)}{f(z)}dz \qquad (1)$$

Now, $f(z)$ has pole of order *p* at $z = a$, so

$$f(z) = \frac{g(z)}{(z-a)^p} , \qquad (2)$$

where *g(z)* is analytic and non-zero within and on γ_1. Taking logarithm of (2) and differentiating, we

$$\log f(z) = \log g(z) - p\log(z-a)$$

Differentiatingw.r.t. z, we *get*

get *i.e.*, $\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} - \frac{p}{z-a}$.

On integrating along γ_1, *we have*

$$\int_{\gamma_1} \frac{f'(z)}{f(z)}dz = \int_{\gamma_1} \frac{g'(z)}{g(z)}dz - p\int_{\gamma_1} \frac{dz}{z-a} \qquad (3)$$

Since $\frac{g'(z)}{g(z)}$ is analytic within and on γ_1, by Cauchy theorem,

$$\int_{\gamma_1} \frac{g'(z)}{g(z)}dz = 0.$$

Thus, (3) gives $\int_{\gamma_1} \frac{f'(z)}{f(z)} dz = -2\pi i p$ (4)

Again, $f(z)$ has a zero of order n at $z = b$, so we can write

$$f(z) = (z-b)^n \phi(z), \quad (5)$$

where $\phi(z)$ is analytic and non-zero within and on T_1.

Taking logarithm, then differentiating, we get

$$\frac{f'(z)}{f(z)} = \frac{n}{z-b} + \frac{\phi'(z)}{\phi(z)}$$

$$\Rightarrow \int_{T_1} \frac{f'(z)}{f(z)} dz = n \int_{T_1} \frac{dz}{z-b} + \int_{T_1} \frac{\phi'(z)}{\phi(z)} dz \quad (6)$$

Since $\frac{\phi'(z)}{\phi(z)}$ is analytic within and on T_1, therefore $\int_{T_1} \frac{\phi'(z)}{\phi(z)} dz = 0$ and thus (6) becomes

$$\int_{T_1} \frac{f'(z)}{f(z)} dz = 2\pi i n \quad (7)$$

Writing (1) with the help of (4) and (7), we get

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = -p + n = n - p \quad (8)$$

Now, suppose that $f(z)$ has poles of order p_m at $z = a_m$ for $m = 1, 2, \ldots, r$ and zeros of order n_m at $z = b_m$ for $m = 1, 2, \ldots, s$ within C. We enclose each pole and zero by circles $\gamma_1, \gamma_2, \ldots, \gamma_r$ and $T_1, T_2, \ldots, T_s$. Thus, (8) becomes

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = \sum_{m=1}^{s} n_m - \sum_{m=1}^{r} p_m$$

Taking $\sum_{m=1}^{s} n_m = N$, $\sum_{m=1}^{r} p_m = P$, we obtain

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = N - P.$$ Which proves the theorem.

3.5.3 Theorem (Argument Principle) : If $f(z)$ is analytic inside and on a closed contour C and does not vanish on C. Then,

$$N = \frac{1}{2\pi} \Delta_C [\arg f(z)],$$

where N stands for the total number of zeros of $f(z)$ inside C and Δ_C represents the variation (change) of $\log f(z)$ around the contour C.

Proof: We know that $\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = N - P$ (1)

where N and P are respectively the total number of zeros and number of poles of $f(z)$ inside C. In the present case, $f(z)$ is analytic and has no pole inside C, therefore $P = 0$. Hence, equation (1) becomes

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N \Rightarrow \int_C \frac{f'(z)}{f(z)}dz = 2\pi iN$$

$$\Rightarrow \qquad 2\pi iN = [\log f(\mathrm{z})]_c = \Delta_C[\log f(\mathrm{z})] \qquad (2)$$

Where Δ_C stands for the variation of $\log f(\mathrm{z})$ as z moves once around the contour C.

Since $\quad z = x + iy = re^{i\theta}$

$\Rightarrow \qquad \log z = \log r + i\theta$

$\Rightarrow \qquad \log z = \log r + i\arg f(z)$

Therefore, we have

$$\log f(z) = \log|f(z)| + i\arg f(z) \ ,$$

Hence, $\Delta_C[\log f(z)] = \Delta_C\left[\log|f(z)|\right] + i\ \Delta_C[\arg f(\mathrm{z})]$

But $\Delta_C\left[\log|f(z)|\right] = 0$. Since $\log|f(z)|$ is single valued i.e., it remains as it is, as z goes once around C. Thus, we get

$$\Delta_C[\log f(z)] = i\ \Delta_C[\arg f(\mathrm{z})].$$

Therefore, from equation (2), we obtain

$$2\pi iN = i\Delta_C[\arg f(\mathrm{z})]$$

$$\Rightarrow \ N = \frac{1}{2\pi}\Delta_C[\arg f(z)]$$

Which proves the theorem.

3.5.4 Rouche's Theorem: If $f(z)$ and $g(z)$ are analytic inside and on a closed contour C and $|g(z)| < |f(z)|$ on C, then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside C.

Proof: First we prove that neither $f(z)$ nor $f(z) + g(z)$ has a zero on C. If $f(z)$ has a zero at $z = a$ on C, then $f(\mathrm{a}) = 0$

Thus,

$$|g(z)| < |f(z)| \ \Rightarrow |g(a)| < |f(a)| = 0$$

$$\Rightarrow \ g(a) = 0 \ \Rightarrow |f(a)| = |g(a)|$$

i.e. $|f(\mathrm{z})| = |\mathrm{g}(\mathrm{z})|$ at $z = a$,

which is contrary to the assumption that $|g(z)| < |f(z)|$ on C.

Again, if $f(z) + g(z)$ has a zero at $z = b$ on C,

then $f(b) + g(b) = 0 \ \Rightarrow f(b) = -g(b)$

i.e. $|f(b)| = |g(b)|$

Thus, neither $f(z)$ nor $f(z) + g(z)$ has a zero on *C*.

Now, let *N* and N' be the number of zeros of $f(z)$ and $f(z) + g(z)$ respectively inside *C*. We are to prove that $N = N'$. Since $f(z)$ and $f(z) + g(z)$ both are analytic within and on *C* and have no pole inside *C*, therefore by the argument principle

$$\frac{1}{2\pi i}\int_C \frac{f'}{f}dz = N - P, \; with \; P = 0, \; gives$$

$$\frac{1}{2\pi i}\int_C \frac{f'}{f}dz = N, \quad \frac{1}{2\pi i}\int_C \frac{f'+g'}{f+g}dz = N'$$

Subtracting these two results, we get

$$\frac{1}{2\pi i}\int_C \left[\frac{f'+g'}{f+g} - \frac{f'}{f}\right]dz = N' - N \qquad (1)$$

Let us take

$$\phi(z) = \frac{g(z)}{f(z)}, \; so \; that \; g = f\phi$$

$$Now, \; |g| < |f| \Rightarrow \left|\frac{g}{f}\right| < 1 \; i.e., \; |\phi| < 1$$

$$\therefore \quad \frac{f'+g'}{f+g} = \frac{f'+f'\phi+f\phi'}{f+f\phi} = \frac{f'(1+\phi)+f\phi'}{f(1+\phi)}$$

$$= \frac{f'}{f} + \frac{\phi'}{1+\phi}$$

$$i.e., \quad \frac{f'+g'}{f+g} - \frac{f'}{f} = \frac{\phi'}{1+\phi} \qquad (2)$$

Using (2) in (1), we get

$$N' - N = \frac{1}{2\pi i}\int_C \frac{\phi'}{1+\phi}dz = \frac{1}{2\pi i}\int_C \phi'(1+\phi)^{-1}dz \qquad (3)$$

Since we have observed that $|\phi| < 1$, so binomial expansion of $(1 + \phi)^{-1}$ is possible and this expansion in powers of ϕ is uniformly convergent and hence term by term integration is possible.

Thus,

$$\int_C \phi'(1+\phi)^{-1}dz = \int_C \phi'(1-\phi+\phi^2-\phi^3+\ldots)dz$$

$$= \int_C \phi' dz - \int_C \phi\phi' dz + \int_C \phi^2\phi' dz + \ldots\ldots.. \qquad (4)$$

Now, the functions *f* and *g* both are analytic within and on *C* and $f \neq 0, g \neq 0$ for any point on *C*, therefore $\phi = \frac{g}{f}$ is analytic and non-zero for any point on *C*. Thus, ϕ and it's all derivatives are analytic and so by Cauchy's theorem, each integral on R.H.S. of (4) vanishes. Thus

$$\int_C \phi'(1+\phi)^{-1}\,dz = 0$$

and therefore from (3), we conclude $N' - N = 0$ i.e., $N = N'$. Which proves the theorem.

3.5.5 Example: Determine the number of roots of the equation

$$z^8 - 4z^5 + z^2 - 1 = 0$$

that lie inside the circle $|z| = 1$.

Solution: Let *C* be the circle defined by $|z| = 1$. Let us take $f(z) = z^8 - 4z^5$, $g(z) = z^2 - 1$.

On the circle *C*,

$$\left|\frac{g(z)}{f(z)}\right| = \left|\frac{z^2-1}{z^8-4z^5}\right| \le \frac{|z|^2+1}{|z|^5\,|4-z^3|}$$

$$\le \frac{1+1}{4-|z|^3} = \frac{2}{4-1} = \frac{2}{3} < 1$$

Thus $|g(z)| < |f(z)|$ and both $f(z)$ and $g(z)$ are analytic within and on *C*, Rouche's theorem implies that the required number of roots is the same as the number of roots of the equation $z^8 - 4z^5 = 0$ in the region $|z| < 1$. Since $z^3 - 4 \neq 0$ for $|z| < 1$, therefore the required number of roots is found to be 5

3.5.6 Example: Determine the number of roots of the equation

$$z^7 - 4z^3 + z - 1 = 0$$

that lie inside the circle $|z| = 1$.

Solution: Let *C* be the circle defined by $|z| = 1$. Let us take $f(z) = -4z^3$, $g(z) = z^7 + z - 1$.

On the circle *C*,

$$\left|\frac{g(z)}{f(z)}\right| = \left|\frac{z^7+z-1}{-4z^3}\right| \le \frac{|z|^7+|z|+1}{4|z|^3} \le \frac{1+1+1}{4(1)} = \frac{3}{4} < 1 .$$

Thus, $|g(z)| < |f(z)|$ and both $f(z)$ and $g(z)$ are analytic within and on *C*. Rouche's theorem implies that the required number of roots is the same as the number of roots of the equation $f(z) = 0$ in the region $|z| < 1$. Since $f(z)$ has three zeros, counting multiplicities, inside the circle $|z| = 1$, so does $f(z)+g(z)$. Thus, the given equation has three roots inside the circle $|z| = 1$.

3.5.7 Example: Prove that all the roots of the equation

$$z^7 - 5z^3 + 12 = 0$$

lie between the circles $|z| = 1$and $|z| = 2$.

Solution:(i) Consider the circle C_1 defined by $|z| = 1$ and suppose that $f(z) = 12$ and $g(z) = z^7 - 5z^3$. Thus, $f(z)$ and $g(z)$ are both analytic within and on C_1.

Also,

$$\left|\frac{g(z)}{f(z)}\right| = \left|\frac{z^7 - 5z^3}{12}\right| \leq \frac{|z|^7 + 5|z|^3}{12} \leq \frac{1+5}{12} = \frac{6}{12} = \frac{1}{2} < 1$$

$$\Rightarrow \left|\frac{g(z)}{f(z)}\right| < 1$$

Thus, $|g(z)| < |f(z)|$ and both $f(z)$ and $g(z)$ are analytic within and on C_1. Hence, by Rouche's theorem, $f(z) + g(z)$ i.e., $z^7 - 5z^3 + 12$ has the same number of zeros inside C_1 as $f(z)$. But $f(z)$ has no zero inside C_1, which implies that $z^7 - 5z^3 + 12 = 0$ has no zero inside C_1.

(ii) Consider the circle C_2 defined by $|z| = 2$ and suppose that $f(z) = z^7$ and $g(z) = -5z^3 + 12$. Thus, $f(z)$ and $g(z)$ are both analytic within and on C_2. Also,

$$\left|\frac{g(z)}{f(z)}\right| = \left|\frac{-5z^3 + 12}{z^7}\right| \leq \frac{5|z|^3 + 12}{|z|^7} \leq \frac{5(2)^3 + 12}{2^7} = \frac{52}{128} < 1$$

$$\Rightarrow \left|\frac{g(z)}{f(z)}\right| < 1.$$

Thus, $|g(z)| < |f(z)|$ and both $f(z)$ and $g(z)$ are analytic within and on C_2. Hence, by Rouche's theorem, $f(z) + g(z)$ i.e., $z^7 - 5z^3 + 12$ has the same number of zeros inside C_2 as $f(z)$. Since $f(z) = z^7$ has 7 zeros inside C_2, therefore $z^7 - 5z^3 + 12 = 0$ also has 7 zeros inside C_2. From these two results, we conclude that the given equation has all its roots between C_1 and C_2.

3.5.8 Example: Use Rouche's theorem to prove that the equation $e^z = az^n$ has n roots inside the circle $|z| = 1$, where $a > e$.

Solution: Let C denotes the circle $|z| = 1$. Let us take $f(z) = az^n$ and $g(z) = -e^z$. So, the given equation is of the form $f(z) + g(z) = 0$. We note that $f(z)$ and $g(z)$ are both analytic within and on C. Further,

$$\left|\frac{g(z)}{f(z)}\right| = \left|\frac{-e^z}{az^n}\right| = \frac{|e^z|}{|a||z^n|} = \frac{\left|1+z+\dfrac{z^2}{2!}+\ldots\right|}{|a||z^n|}$$

$$\leq \frac{1+|z|+\dfrac{|z^2|}{2!}+\ldots}{a|z|^n}$$

$$= \frac{1+\dfrac{1}{1!}+\dfrac{1}{2!}+\dfrac{1}{3!}+\ldots}{a} = \frac{e}{a} < 1 \quad (\because a > e)$$

$$\Rightarrow \ |g(z)| < |f(z)| \ .$$

Hence, by Rouche's theorem, $f(z)$ and $f(z)+g(z)$ have same number of zeros inside $|z|=1$. But $f(z)$ has *n*- zeros and consequently, the given equation has *n* roots inside $|z|=1$.

3.5.9 Theorem (Fundamental Theorem of Algebra): Every polynomial of degree *n* has exactly *n* zeros.

Proof: Let us consider the polynomial $a_0+a_1z+a_2z^2+\ldots+a_nz^n, \quad a_n \neq 0$

We take $f(z)=a_nz^n, \ \ g(z)=a_0+a_1z+a_2z^2+\ldots+a_{n-1}z^{n-1}$

Let *C* be a circle $|z| = r$, where $r > 1$

$$|f(z)| = |a_nz^n| = |a_n|r^n$$

Now,

$$|g(z)| \leq |a_0|+|a_1|r+|a_2|r^2+\ldots.+|a_{n-1}|r^{n-1}$$

$$\leq (|a_0|+|a_1|+|a_2|+\ldots.+|a_{n-1}|)r^{n-1}$$

Therefore,

$$\left|\frac{g(z)}{f(z)}\right| \leq \frac{\left(|a_0|+|a_1|+\ldots+|a_{n-1}|\right)r^{n-1}}{|a_n|r^n}$$

$$\frac{\left(|a_0|+|a_1|+\ldots+|a_{n-1}|\right)}{|a_n|r}$$

Hence, $|g(z)| < |f(z)|$, provided that

$$\frac{\left(|a_0|+|a_1|+\ldots+|a_{n-1}|\right)}{|a_n|r} < 1$$

$$i.e., \ r > \frac{\left(|a_0|+|a_1|+\ldots+|a_{n-1}|\right)}{|a_n|} \tag{1}$$

Since *r* is arbitrary, therefore we can choose *r* large enough so that (1) is satisfied. Now, applying Rouche's theorem, we find that the given polynomial $f(z)+g(z)$ has the same number of zeros as $f(z)$.

But $f(z)$ has exactly n zeros all located at $z = 0$. Hence, the given polynomial has exactly n zeros.

3.6 Inverse Function: If $f(z) = w$ has a solution $z = F(w)$, then we may write $f\{F(w)\} = w$, $F\{f(z)\} = z$. The function F defined in this way, is called inverse function of f.

3.6.1 Theorem (Inverse Function Theorem): Let a function $w = f(z)$ be analytic at a point $z = z_0$ where $f'(z_0) \neq 0$ and $w_0 = f(z_0)$. Then there exists a neighbourhood of w_0 in the w-plane in which the function $w = f(z)$ has a unique inverse $z = F(w)$ in the sense that the function F is single-valued and analytic in that neighbourhood such that $F(w_0) = z_0$ and $F'(w) = \dfrac{1}{f'(z)}$.

Proof: Consider the function $f(z) - w_0$. By hypothesis, $f(z_0) - w_0 = 0$. Since $f'(z_0) \neq 0$, f is not a constant function and therefore, neither $f(z) - w_0$ nor $f'(z)$ is identically zero. Also $f(z) - w_0$ is analytic at $z = z_0$ and so it is analytic in some neighbourhood of z_0. Again, since zeros are isolated, neither $f(z) - w_0$ nor $f'(z)$ has any zero in some deleted neighbourhood of z_0. Hence, there exists $\varepsilon > 0$ such that $f(z) - w_0$ is analytic for $|z - z_0| \leq \varepsilon$ and

$$f(z) - w_0 \neq 0,\ f'(z) \neq 0 \text{ for } 0 < |z - z_0| \leq \varepsilon \ .$$

Let D denote the open disc

$$\{z : |z - z_0| < \varepsilon\}$$

and C denotes its boundary

$$\{z : |z - z_0| = \varepsilon\}.$$

Since $f(z) - w_0$ for $|z - z_0| \leq \varepsilon$, we conclude that $|f(z) - w_0|$ has a positive minimum on the circle C. Let

$$\min_{z \in C} |f(z) - w_0| = m \text{ and choose } \delta \text{ such that } 0 < \delta < m.$$

We now show that the function $f(z)$ assumes exactly once in D every value w_1 in the open disc $T = \{w : |w - w_0| < \delta\}$. We apply Rouche's theorem to the functions $w_0 - w_1$ and $f(z) - w_0$. The condition of the theorem are satisfied, since

$$|w_0 - w_1| < \delta < m = \min_{z \in C} |f(z) - w_0| \leq |f(z) - w_0| \ on\ C.$$

Thus, we conclude that the functions

$$f(z) - w_0 \text{ and } (f(z) - w_0) + (w_0 - w_1) = f(z) - w_1,$$

have the same number of zeros in D. But the function $f(z) - w_0$ has only one zero in D i.e. a simple zeros at z_0, since

$$(f(z) - w_0)' = f'(z) \neq 0 \ \text{ at } z_0.$$

Hence, $f(z) - w_1$ must also have only one zero, say z_1 in D. This means that the function $f(z)$ assumes the value w, exactly once in D. It follows that the function $w = f(z)$ has a unique inverse, say $z = F(w)$ in D such that F is single-valued and $w = f\{F(w)\}$. We now show that the function F is analytic in D. For fix w_1 in D, we have $f(z) = w_1$ for a unique z_1 in D. If w is in T and $F(w) = z$, then

$$\frac{F(w)-F(w_1)}{w-w_1} = \frac{z-z_1}{f(z)-f(z_1)} \tag{1}$$

It is noted that T is continuous. Hence, $z \to z_1$ whenever $w \to w_1$. Since $z_1 \in D$, as shown above $f'(z_1)$ exists and is zero. If we let $w \to w_1$, then (1) shows that

$$F'(w_1) = \frac{1}{f'(z_1)}.$$

Thus, $F'(w)$ exists in the neighbourhood T of w_0 so that the function F is analytic there.

SECTION-IV

4.1 Calculus of Residues: We know that in the neighbourhood of an isolated singularity z=z_0, a one valued analytic function $f(z)$ may be expanded in a Laurent series as

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

The coefficient b_1 of $\dfrac{1}{(z-z_0)}$ in the Laurent series is called the residue of the function $f(z)$ at the isolated singularity z_0. We shall use the notation

$b_1 = \mathrm{Re}\,s(f(z), z_0) = \mathrm{Re}\,s(z = z_0)$ to denote the residue of f at z_0.

Therefore, $\mathrm{Re}\,s(f(z), z_0) = b_1 = \dfrac{1}{2\pi i}\int_{\Upsilon} f(z)\,dz$,

where Υ is any circle with centre $z = z_0$, which excludes all other singularities of $f(z)$.

4.1.1 Computation of Residues in some special cases:

4.1.1.1 Residue at a Simple Pole: (i) If f has a simple pole at $z = z_0$, then $Res(f(z), z_0) = \lim\limits_{z \to z_0}(z - z_0) f(z)$.

Proof: Since f has a simple pole at $z = z_0$, its Laurent expansion convergent in annulus $0 < |z - z_0| < R$ has the form

$$f(z) = \frac{b_1}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots$$

Where $b_1 \neq 0$. By multiplying both sides of this series by $(z - z_0)$ and then taking the limit as $z \to z_0$, we obtain

$$\lim_{z\to z_0}(z-z_0)f(z) = \lim_{z\to z_0}[b_1 + a_0(z-z_0) + a_1(z-z_0)^2 + \ldots] = b_1 = Res(f(z), z_0).$$

(ii) If f has a simple pole at $z = z_0$ and $f(z)$ is of the form $f(z) = \dfrac{\varphi(z)}{\psi(z)}$ i.e. a rational function then

$$Res(f(z), z_0) = \lim_{z\to z_0}(z-z_0) f(z) = \lim_{z\to z_0}(z-z_0)\frac{\varphi(z)}{\psi(z)}$$

$$= \lim_{z\to z_0} \frac{\varphi(z)}{\dfrac{\psi(z)-\psi(z_0)}{(z-z_0)}} = \frac{\varphi(z_0)}{\psi'(z_0)}$$

where $\psi(z_0) = 0$, $\psi'(z_0) \neq 0$, since $\psi(z)$ has a simple zero at z= z_0.

4.1.1.2 Residue at a Pole of Order n: If f has a pole of order n at z = z_0, then

$$Res(f(z), z_0) = \frac{1}{(n-1)!}\lim_{z\to z_0}\frac{d^{n-1}}{dz^{n-1}}\left[(z-z_0)^n f(z)\right].$$

Proof: Because f is assumed to have pole of order n at $z = z_0$, its Laurent expansion is convergent inannulus $0 < |z - z_0| < R$ and must have the form

$$f(z) = \frac{b_n}{(z - z_0)^n} + \ldots + \frac{b_2}{(z - z_0)^2} + \frac{b_1}{(z - z_0)^1} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots$$

where $b_n \neq 0$. We multiply the last expression by $(z - z_0)^n$,

$$(z - z_0)^n f(z) = b_n + \ldots + b_2(z - z_0)^{n-2} + b_1(z - z_0)^{n-1} + a_0(z - z_0)^n + a_1(z - z_0)^{n+1} + \ldots$$

and then differentiate both sides of the equality *(n-1)* times, we have

$$\frac{d^{n-1}}{dz^{n-1}}\left[(z - z_0)^n f(z)\right] = (n-1)!b_1 + n!a_0(z - z_0) + \ldots$$

Since all the terms on the right hand side after the first involve positive integer powers of (z-z_0), taking the limit as z tends to z_0, we get

$$\lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}}[(z - z_0)^n f(z)] = (n-1)!b_1$$

$$Res(f(z), z_0) = b_1 = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}}[(z - z_0)^n f(z)],$$

which is the required result.

4.1.1.3 Residue at infinity: If f(z) is analytic or has an isolated singularity at infinity and if C is a circle enclosing all its singularities in the finite parts of the *z-plane*, the residue of f(z) at infinity is defined by

$$Res(f(z), \infty) = \frac{-1}{2\pi i} \int_C f(z)\,dz = -\text{Coefficient of } b_1 \text{ in expansion of function } f(z).$$

Also, Res($f(z)$,∞) = – {Sum of residues at all finite singularities}

4.1.2 Remarks: (i) The function may be regular at infinity, yet has a residue there.

For example, consider a function $f(z) = \dfrac{b}{z - a}$.

Since, $Res(f(z), \infty) = \dfrac{-1}{2\pi i} \displaystyle\int_C f(z)\,dz = \dfrac{-1}{2\pi i} \int_C \dfrac{b}{z - a}\,dz$

$$= \frac{-b}{2\pi i} \int_0^{2\pi} \frac{re^{i\theta} i}{re^{i\theta}}\,d\theta = \frac{-b}{2\pi} \int_0^{2\pi} d\theta = -b,$$

where C being the circle $|z - a| = r$.

Also, $z = a$ is a simple pole of $f(z)$ and its residue there is $\dfrac{1}{2\pi i} \displaystyle\int_C f(z)\,dz = b$.

Thus, Res($f(z)$,∞) = $-$Res($f(z)$,a) $= -b$

(ii) If the function is analytic at a point $z = a$, then its residue at $z = a$ is zero but not so at infinity.

(iii) In the definition of residue at infinity, C may be any closed contour enclosing all the singularities in the finite parts of the z-plane.

4.1.3 Example: Find the residue of $\frac{z^4}{z^2+a^2}$ at $z = -ia$.

Solution: Let $f(z)=\frac{z^4}{z^2+a^2}$

Poles of $f(z)$ are $z = \pm ia$. Thus $z = -ia$ is a simple pole.

$$\text{So, Res }(f(z), -ia) = \lim_{z\to -ia}\left(z+ia\right)f\left(z\right)$$

$$= \lim_{z\to -ia}(z+ia)\frac{z^4}{(z+ia)(z-ia)}$$

$$= \lim_{z\to -ia}\frac{z^4}{\left(z-ia\right)} = \frac{(-ia)^4}{-2ia} = \frac{ia^3}{2}.$$

4.1.4 Example: Find the residue of $e^{iz}z^{-4}$ at its poles.

Solution: Let $f(z)=\frac{e^{iz}}{z^4}$.Then, $f(z)$ has a pole of order 4 at $z = 0$. So,

$$\text{Res}(f(z),0) = \frac{1}{3!}\lim_{z\to 0}\frac{d^3}{dz^3}e^{iz} = \frac{-i}{6}.$$

Alternatively, by the Laurent's expansion, we can find residue of f(z) as the negative of coefficient of 1/z.

4.1.5 Example: Find the residue of $\frac{z^3}{(z^2-1)}$ at $z = \infty$.

Solution: Since $z = 1, -1$ are the simple poles. Thus,

$$Res(f(z),1) = \lim_{z\to 1}\left(z-1\right)f\left(z\right)$$

$$= \lim_{z\to 1}\frac{z^3}{z+1} = \frac{1}{2}$$

$$Res(f(z),-1) = \lim_{z\to -1}\left(z+1\right)f\left(z\right)$$

$$= \lim_{z\to -1}\frac{z^3}{z-1} = \frac{1}{2}$$

As we know,

$$Res(f(z),\infty) = -[Res(f(z),1) + Res(f(z),-1)]$$

$$= -\left\{\frac{1}{2}+\frac{1}{2}\right\} = -1.$$

Alternatively, by the Laurent's expansion, we can find residue of f(z) as the negative of coefficient of 1/z.

4.1.6 Example: Find the residue of $\dfrac{z^3}{(z-1)^4(z-2)(z-3)}$ at its pole.

Solution: Let $f(z)=\dfrac{z^3}{(z-1)^4(z-2)(z-3)}$. Poles of $f(z)$ are given by $z = 1,2,3$; where $z = 1$ is a pole of order 4 and $z = 2$ and $z = 3$ are simple poles.

Therefore, Res($f(z)$,1) $=\dfrac{\varphi^3(1)}{3!}$ Where $f(z)=\dfrac{\varphi(z)}{(z-1)^4}$ i.e. $\varphi(z)=\dfrac{z^3}{(z-2)(z-3)}$

Resolve $\varphi(z)$ into partial fractions, we get

$$\varphi(z) = (z + 5) - \frac{8}{(z-2)} + \frac{27}{(z-3)}$$

$$\varphi'(z) = 1 + \frac{8}{(z-2)^2} - \frac{27}{(z-3)^2}$$

$$\varphi''(z) = \frac{-16}{(z-2)^3} + \frac{54}{(z-3)^3}$$

$$\varphi'''(z) = \frac{48}{(z-2)^4} - \frac{162}{(z-3)^4}$$

$$\varphi'''(1) = \frac{303}{8}$$

$$\text{Res}(f(z),1) = \frac{303}{8(6)} = \frac{101}{16}$$

$$\text{Res}(f(z),2) = \lim_{z\to 2}(z-2)f(z) = \lim_{z\to 2}\frac{z^3}{(z-1)^4(z-3)} = -8$$

$$\text{Res}(f(z),3) = \lim_{z\to 3}(z-3)f(z) = \lim_{z\to 3}\frac{z^3}{(z-1)^4(z-2)} = \frac{27}{16}.$$

4.1.7 Example: Find the residue of $f(z) = \dfrac{\pi \cot \pi z}{z^2}$.

Solution: Here $f(z) = \dfrac{\pi \cos \pi z}{z^2 \sin \pi z}$.

We note that $f(z)$ has a double pole at $z = 0$ and simple pole at $z = n;\ n = \pm 1, \pm 2, \pm 3, \ldots$

Therefore, $$Res(f(z), n) = \left[\frac{\dfrac{\pi \cos \pi z}{z^2}}{\pi \cos \pi z}\right]_{z=n} = \frac{1}{n^2}$$

Also, $$f(z)=\frac{\pi\cos\pi z}{z^2\sin\pi z}=\frac{\pi\left(1-\frac{\pi^2z^2}{2!}+o(\mathrm{z}^4)\right)}{z^2\left(\pi z-\frac{\pi^3z^3}{3!}+o(\mathrm{z}^5)\right)}$$

$$=\frac{\pi\left[1-\frac{\pi^2z^2}{2!}+o(\mathrm{z}^4)\right]}{z^2\pi z\left[1-\frac{\pi^2z^2}{3!}+o(\mathrm{z}^4)\right]}$$

$$=\frac{1}{z^3}\left[1-\frac{\pi^2z^2}{2!}+o(\mathrm{z}^4)\right]\left[1-\left(\frac{\pi^2z^2}{3!}-o(\mathrm{z}^4)\right)\right]^{-1}$$

$$=\frac{1}{z^3}\left[1-\frac{\pi^2z^2}{2!}+o(\mathrm{z}^4)\right]\left[1+\frac{\pi^2z^2}{3!}-o(\mathrm{z}^4)+\ldots\right]$$

$$=\frac{1}{z^3}\left[Coeff.of\ z^2+\ldots\right]$$

$$Coeff.of\ z^2=\frac{-3\pi^2z^2}{2!}+\frac{\pi^2z^2}{3!}=-\frac{\pi^2z^2}{3}$$

Also, by definition, *Res(f(z),0) = Coefficient of* $\frac{1}{z}=\frac{-\pi^2}{3}$.

4.1.8 Exercise: Use an appropriate Laurent series to find the residue of the following:

(i) $f(z)=\frac{2}{(z-1)(z+4)}$, Res($f$(z), 1)

(ii) $f(z)=\frac{1}{z^3(1-z)^3}$, Res($f$(z), 0)

(iii) $f(z)=\frac{4z-6}{z(2-z)}$, Res(f(z), 0)

(iv) $f(z)=(z+3)^2\sin\left(\frac{2}{z+3}\right)$, Res($f$(z), -3)

(v) $f(z)=e^{-2/z^2}$, Res(f(z), 0)

(vi) $f(z)=\frac{e^{-z}}{(z-2)^2}$, Res($f$(z), 2)

4.1.9 Exercise: find residues at each pole of the given function

(i) $f(z)=\frac{z}{z^2+16}$ (ii) $f(z)=\frac{4z+8}{2z-1}$

(iii) $f(z)=\frac{1}{z^4+z^3-2z^2}$ (iv) $f(z)=\frac{1}{(z^2-2z+2)^2}$

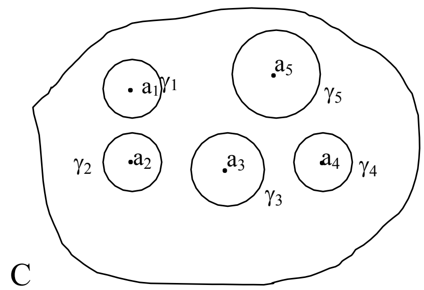
(v) $f(z)=\frac{\cos z}{z^2(z-\pi)^3}$ (vi) $f(z)=\frac{e^z}{e^z-1}$

(vi) $f(z)=\sec z$

4.1.10Theorem (Cauchy Residue Theorem): Let f(z) be one-valued and analytic inside and on a simple closed contour C, except for a finite number of poles within C.Then

$$\int_C f(z)\,dz = 2\pi i[\text{Sum of residues of f(z) at its poles within C}].$$

Proof: Let $a_1, a_2, \ldots, a_n$ be the poles of f(z) inside C. Draw a set of circles Υ_r of radii ε and centre a_r $(r = 1, 2, \ldots, n)$ which do not overlap and all lie within C. Then f(z) is regular in the domain bounded externally by C and internally by the circlesΥ_r.



Then by cor. to Cauchy's theorem, we have

$$\int_C f(z)\,dz = \sum_{r=1}^{n} \int_{\Upsilon_r} f(z)\,dz\,.$$

Now if $a_r$is a pole of order m, then by Laurent's theorem, f(z) can be expressed as

$$f(z) = \Phi(z) + \sum_{s=1}^{m} \frac{b_s}{(z-a_r)^s}\,,$$

where Φ(z) is regular within and on Υ_r . Then,

$$\int_{\Upsilon_r} f(z)\,dz = \sum_{s=1}^{m} \int \frac{b_s}{(z-a_r)^s}\,dz$$

where $\int_{\Upsilon_r} \Phi(z)\,dz = 0$ by Cauchy's theorem.

Now, on Υ_r we have, $|z-a_r| = \in$ i.e. $z = a_r + \in e^{i\theta}$

Thus, $dz = \in ie^{i\theta} d\theta,$where θ varies from 0 to 2π as the point z moves once round Υ_r .

$$\int_{\Upsilon_r} f(z)\,dz = \sum_{s=1}^{m} b_s \in^{1-s} \int_0^{2\pi} e^{(1-s)i\theta} i\,d\theta$$

$$= 2\pi i b_1$$

$$= 2\pi i\left[Residue\ of\ f\left(z\right)\ at\ a_r\right], \text{ where} \int_0^{2\pi} e^{(1-s)i\theta} d\theta = \begin{cases} 0, if\ s \neq 1 \\ 2\pi, if\ s = 1 \end{cases}$$

Hence, from (1), $\int_C f(z)\,dz = \sum_{r=1}^{n} 2\pi i\left[Residue\,of\ f(z)\,at\,a_r\right]$

$$= 2\pi i \sum_{r=1}^{n} \left[Residue\,of\ f(z)\,at\,a_r\right]$$

$= 2\pi i$ [Sum of residues of $f(z)$ at its poles inside C].

4.2 Evaluation of Real Trigonometric integral: In this section, we shall see how residue theory can be used to evaluate real integrals of the following forms:

(i) $\int_0^{2\pi} F(cos\theta, sin\theta)d\theta$

(ii) $\int_{-\infty}^{\infty} f(x)dx$

(iii) $\int_{-\infty}^{\infty} cos\alpha x\, dx$

(iv) $\int_{-\infty}^{\infty} sin\alpha x\, dx$,

where F and *f* are rational functions. For the rational function *f*(x) = p(x)/q(x), we will assume that the polynomial p and q have no common factors.

Integrals of the form $\boldsymbol{\int_0^{2\pi} F(cos\theta, sin\theta)d\theta}$: The basic idea here is to convert a real trigonometric integral into a complex integral, where the contour C is the unit circle $|z| = 1$ centered at the origin. To do this, we parametrize this contour by z = $e^{i\theta}$, $0 \leq \theta \leq 2\pi$.

We can then write

$$dz = ie^{i\theta}d\theta\ , \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}.$$

Since dz = i$e^{i\theta}d\theta$ = iz $d\theta$ and $z^{-1} = \frac{1}{z} = e^{-i\theta}$, these three quantities are equivalent to

$$d\theta = \frac{dz}{iz}, \cos\theta = \frac{z+z^{-1}}{2}\ , \sin\theta = \frac{z-z^{-1}}{2}.$$ Thus, we get

$$\int_0^{2\pi} F\left(cos\theta, sin\theta\right)d\theta = \int_C F\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2}\right)\frac{dz}{iz},$$ where C is the unit circle $|z| = 1$.

4.2.1 Example: Evaluate $\int_0^{\pi} \frac{a}{a^2 + sin^2\theta}d\theta$, where *a>0.*

Solution: Let $I = \int_0^{\pi} \frac{a}{a^2+sin^2\theta}d\theta = \int_0^{\pi}\frac{2a}{2a^2+2sin^2\theta}d\theta = \int_0^{\pi}\frac{2a}{2a^2+1-cos2\theta}d\theta$

$$= \int_0^{2\pi}\frac{a}{2a^2+1-cost}dt, \{2\theta = t\}$$

$$= \int_0^{2\pi}\frac{a}{2a^2+1-\frac{\left(e^{it}+e^{-it}\right)}{2}}dt$$

Putting $z = e^{it}$, such that $dz = e^{it}idt$, we get

$$I = \int_C \frac{2a}{2(2a^2+1)+\left(z+z^{-1}\right)}\frac{dz}{iz} = 2ai\int_C \frac{1}{z^2-2(2a^2+1)z+1}dz\ .$$

or $$I = 2ai\int_C f(z)dz,$$ where C is unit circle $|z| = 1$.

and $\quad f(z)=\dfrac{1}{z^2-2(2a^2+1)z+1}$

Now the poles are given by $z^2-2(2a^2+1)z+1=0$

i.e. z $=(2a^2+1)\pm 2a\sqrt{(a^2+1)}$

we take $\alpha=(2a^2+1)+2a\sqrt{(a^2+1)}$

$\beta=(2a^2+1)-2a\sqrt{(a^2+1)}$

Thus, the poles are $z=\alpha,\beta$

Clearly, $|\alpha|>1$ and since $|\alpha\beta|=1\Rightarrow|\beta|<1$. Thus, f(z) has only one simple pole z $=\beta$ that lies within C.

$$Res\left(f\left(z\right),\beta\right)=Res\left(z=\beta\right)=\lim_{z\to\beta}(z-\beta)f(z)=\frac{1}{\beta-\alpha}=\frac{-1}{4a\sqrt{(a^2+1)}}$$

Hence, by Cauchy's Residue Theorem

$$\int_C f(z)\,dz=2\pi i \text{ [Sum of residues of } f(z) \text{ at its poles within C]}$$

$$=2\pi i\left(\frac{-1}{4a\sqrt{(a^2+1)}}\right)=\frac{\pi}{\sqrt{(a^2+1)}}.$$

4.2.2 Example: Prove that $\int_0^{2\pi}\dfrac{d\theta}{a+b\cos\theta}=\dfrac{2\pi}{\sqrt{a^2-b^2}}$, a>b>0.

Solution: Let I$=\int_0^{2\pi}\dfrac{d\theta}{a+b\cos\theta}=\int_0^{2\pi}\dfrac{2d\theta}{2a+b(e^{i\theta}+e^{-i\theta})}$

Put $z=e^{i\theta}\Rightarrow dz=ie^{i\theta}d\theta$.

Therefore, $\quad$ I $=\dfrac{2}{ib}\int_C f(z)\,dz$, where $f(z)=\dfrac{1}{z^2+\dfrac{2a}{b}z+1}$, *where C is unit circle* $|z|=1$.

Poles of $f(z)$ are given by z = $\alpha=\dfrac{-a+\sqrt{a^2-b^2}}{b}$ and z= $\beta=\dfrac{-a-\sqrt{a^2-b^2}}{b}$

Since $|\beta|>1\,and\,|\alpha\beta|=1\Rightarrow|\alpha|<1$ $\quad$ [$\because$ a > b > 0]

So, z=α is the only pole lying within C.

$$\text{Res}(z=\alpha)=\lim_{z\to\alpha}(z-\alpha)\,f(z)=\lim_{z\to\alpha}(z-\alpha)\frac{1}{z^2+\dfrac{2az}{b}+1}$$

$$=\lim_{z\to\alpha}\frac{z-\alpha}{(z-\alpha)(z-\beta)}=\frac{1}{\alpha-\beta}=\frac{b}{2\sqrt{a^2-b^2}}$$

Hence, by Cauchy residue theorem,

$$\int_C f(\mathrm{z})\,\mathrm{dz} = 2\pi i \frac{b}{2\sqrt{a^2-b^2}} = \frac{\pi i b}{\sqrt{a^2-b^2}}$$

Therefore, from (1), we get

$$\mathrm{I} = \frac{2}{ib}\frac{i\pi b}{\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

4.2.3 Example: Prove that $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta = \frac{\pi}{6}$.

Solution: Let $\mathrm{I} = \int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta$

$$= \text{real part of } \int_0^{2\pi} \frac{e^{i2\theta}.d\theta}{5+2\left(e^{i\theta}+e^{-i\theta}\right)}$$

$$= \text{real part of } \int_C \frac{z^2}{5+2\left(z+z^{-1}\right)}\frac{dz}{iz}, z = e^{i\theta}$$

$$= \text{real part of } \frac{1}{i}\int_C f(z)dz,$$

where $f(z) = \frac{z^2}{2z^2+5z+2} = \frac{z^2}{(\mathrm{z}+2)(2\,\mathrm{z}+1)}$, where C is unit circle $|z| = 1$.

Poles of $f(z)$ are given by z = -2, $\frac{-1}{2}$. So z = $\frac{-1}{2}$ lies within C.

$$\mathrm{Res}\left(z = \frac{-1}{2}\right) = \lim_{z\to\frac{-1}{2}}\left(z+\frac{1}{2}\right)\frac{z^2}{\left(2z+1\right)(z+2)}$$

$$= \lim_{z\to\frac{-1}{2}}(z+\frac{1}{2})\frac{z^2}{2(z+\frac{1}{2})(z+2)} = \frac{\frac{1}{4}}{\frac{2.3}{2}} = \frac{1}{12}$$

So, using Cauchy residue theorem, we have

$$\mathrm{I} = \text{real part of} \frac{1}{i}.2\pi i.\frac{1}{12} = \frac{\pi}{6}$$

4.2.4 Example: Prove that $\int_0^{2\pi} \frac{\cos n\theta}{1+2a\cos\theta+a^2}.d\theta = \frac{2\pi\left(-1\right)^n a^n}{1-a^2}$ where $a^2 < 1$ and n is a positive integer.

Solution: Let $\mathrm{I} = \int_0^{2\pi} \frac{\cos n\theta}{1+2a\cos\theta+a^2} d\theta$

$$= \text{real part of} \int_0^{2\pi} \frac{e^{in\theta}.d\theta}{1+2a\cos\theta+a^2}$$

$$= \text{real part of} \int_0^{2\pi} \frac{z^n}{1+a(z+z^{-1})+a^2} \cdot \frac{dz}{iz}$$

$$= \text{real part of} \frac{1}{ai} \int_C \frac{z^n dz}{(z+a)(z+\frac{1}{a})}$$

$$= \text{real part of} \frac{1}{ai} \int_C f(z)dz \text{ ,where } f(z) = \frac{z^n}{(z+a)(z+\frac{1}{a})} \text{ , C is unit circle } |z| = 1.$$

Poles of *f(z)* are given by $z = -a,\ z = \frac{-1}{a}$.

So $z = -a$ is a simple pole lying within C, $\left(\because a^2 < 1 \text{ implies } a < 1 \text{ and } \frac{1}{a} > 1\right)$.

$$\therefore \text{Res}(z=-a) = \lim_{z \to -a} (z+a) \frac{z^n}{(z+a)(z+\frac{1}{a})} = \frac{(-1)^n a^{n+1}}{1-a^2}$$

So, by Cauchy residue Theorem

$$\int_C f(z)dz = \frac{2\pi i(-1)^n a^{n+1}}{1-a^2}$$

So, I = real part of $\frac{1}{ai} \cdot \frac{2\pi i(-1)^n a^{n+1}}{1-a^2} = \frac{(-1)^n 2\pi a^n}{1-a^2}$

4.2.5 Example: Prove that $\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta - \mathrm{n}\,\theta)\,\mathrm{d}\,\theta = \frac{2\pi}{n!}$,where n is a +ve integer.

Solution: Let I = $\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta - \mathrm{n}\,\theta)\,\mathrm{d}\,\theta$

$$= \text{real part of} \int_0^{2\pi} e^{\cos\theta} e^{(\sin\theta - \mathrm{n}\theta)\mathrm{i}} d\theta$$

$$= \text{real part of} \int_0^{2\pi} e^{\cos\theta + i\sin\theta - in\theta} d\theta$$

$$= \text{real part of} \int_0^{2\pi} e^{e^{i\theta}} . e^{-in\theta} d\theta$$

$$= \text{real part of} \int_0^{2\pi} e^z z^{-n} . \frac{dz}{iz}, \quad z = e^{i\theta}$$

$$= \text{real part of} \frac{1}{i} \int_C \frac{e^z}{z^{n+1}} .dz$$

$$= \text{real part of} \frac{1}{i} \int_C f(\mathrm{z})\,\mathrm{dz} \text{ , where } f(\mathrm{z}) = \frac{e^z}{z^{n+1}} \text{ ,C is unit circle } |\mathrm{z}| = 1.$$

Clearly, $z = 0$ is a simple pole of order $n+1$ for *f(z),* lying within C.

$$\text{Res}(z=0) = \lim_{z\to 0}\frac{1}{n!}\frac{d^n}{dz^n}\left\{z^{n+1}\frac{e^z}{z^{n+1}}\right\} = \frac{1}{n!}$$

So, by Cauchy Residue Theorem, we have

$$\text{I} = \text{real part of}\,\frac{1}{i}\,2\pi i\,\frac{i}{n!} = \frac{2\pi}{n!}\ .$$

4.2.6 Example: Show that, $\int_0^{\pi}\tan(\theta+\text{ia})\,\text{d}\,\theta = \text{i}\,\pi$ where a is positive real number.

Solution: Let $\text{I} = \int_0^{\pi}\tan(\theta+\text{ia})\,\text{d}\,\theta = \int_0^{\pi}\frac{\sin(\theta+\text{ia})}{\cos(\theta+\text{ia})}d\theta$

$$= \int_0^{\pi}\frac{2\sin(\theta+\text{ia}).\cos(\theta-\text{ia})}{2\cos(\theta+\text{ia}).\cos(\theta-\text{ia})}d\theta$$

$$= \int_0^{\pi}\frac{\sin 2\theta+\sin 2ia}{\cos 2\theta+\cos 2ia}d\theta$$

Put 2θ =t, dθ =dt/2

$$= \int_0^{2\pi}\frac{\sin t+\sin 2ia}{\cos t+\cos 2ia}.\frac{dt}{2}$$

$$= \frac{1}{2}\int_0^{2\pi}\frac{\sin t+i\sinh 2a}{\cos t+\cosh 2a}dt$$

$$= \frac{1}{2}\int_0^{2\pi}\frac{\left(\dfrac{e^{it}-e^{-it}}{2i}\right)+i\,sinh\,2a}{\left(\dfrac{e^{it}+e^{-it}}{2}\right)+cosh\,2a}dt$$

$$= \frac{1}{2i}\int_0^{2\pi}\frac{e^{it}-e^{-it}-2\,sinh\,2a}{e^{it}+e^{-it}+2\,cosh\,2a}.dt$$

$$= \frac{1}{2i}\int_C\frac{\left(z-z^{-1}\right)-2\,sinh\,2a}{\left(z+z^{-1}\right)+2\,cosh\,2a}.\frac{dz}{iz}, z=e^{it}$$

$$= \frac{-1}{2}\int_C\frac{z^2-2z\,sinh\,2a-1}{(z^2+2z\,cosh\,2a+1)\,z}.dz$$

$$= \frac{-1}{2}\int_C f(z)\,dz\text{, where } f(z) = \frac{z^2-2z\,sinh\,2a-1}{z\left(z^2+2z\,cosh\,2a+1\right)}\text{ , C is unit circle } |z|=1.$$

Now, poles of $f(z)$ are given by $z\left(z^2+2z\,cosh\,2a+1\right)=0$.

$$\therefore z=0\ \ and\ \ z=-cosh\,2a\pm sinh\,2a$$

Let $\alpha = -cosh\,2a+sinh\,2a\ ,\ \beta=-cosh\,2\,a-sinh\,2\,a$

So, $f(z)$ has two simple pole $z=0$ and $z=\alpha$ lying within C($|\beta|>1$).

$$Res(z=0)=\lim_{z\to 0}\left(z-0\right)f(z)$$

$$=\lim_{z\to 0} z.\frac{z^2-2z\,sinh\,2a-1}{z\left(z^2+2z\,cosh\,2a+1\right)}=-1$$

and $$Res(z=\alpha)=\lim_{z\to\alpha}(z-\alpha)\,f(z)$$

$$=\lim_{z\to\alpha}\left(z-\alpha\right)\frac{z^2-2z\,sinh\,2a-1}{\left(z-\alpha\right)\left(z-\beta\right).z}$$

$$=\frac{\alpha^2-2\alpha\,sinh\,2a-1}{\alpha\left(\alpha-\beta\right)}=\frac{\alpha-2\,sinh\,2a-\frac{1}{\alpha}}{\alpha-\beta}$$

$$=\frac{\left(\alpha-\beta\right)-2\,sinh\,2a}{\alpha-\beta}=\frac{2\,sinh\,2a-2\,sinh\,2a}{2\,sinh\,2a}=0$$

So, by Cauchy residue theorem, $I=\frac{-1}{2}\left(2\pi i\right)\left[-1+0\right]=\pi i$.

4.2.7 Example: Evaluate $\int_0^{\pi}\frac{\sin^4\theta}{a+b\cos\theta}d\theta$ where *a>b>0.*

Solution: We note that $\sin^4\theta$ is an even function of θ and $sin\theta$ and $cos\theta$ are periodic functions of the period 2π. Hence, the given integral can be written as $I=\frac{1}{2}\int_0^{2\pi}\frac{\sin^4\theta}{a+b\cos\theta}d\theta$

Therefore, $$I=\frac{1}{2}\int_C\frac{\left(\frac{z-z^{-1}}{2i}\right)^4}{a+b\left(\frac{z+z^{-1}}{2}\right)}.\frac{dz}{iz},\ z=e^{i\theta},$$

$$=\frac{1}{16ib}\int_C\frac{\left(z^2-1\right)^4}{z^4\left(z^2+\frac{2az}{b}+1\right)}.dz$$

$$=\frac{1}{16ib}\int_C f(z)\,dz$$,where $f(z)=\frac{(z^2-1)^4}{z^4\left(z^2+\frac{2az}{b}+1\right)}$ and C being a unit circle.

Now, the poles of $f(z)$ are z = 0 of order 4 and $z=\alpha,\beta$, where $\alpha=\frac{-a+\sqrt{a^2-b^2}}{b}$ *and* $\beta=\frac{-a-\sqrt{a^2-b^2}}{b}$.

Now $\left|\alpha\right|<1, \left|\beta\right|>1$ $\{\because a>b>0\}$

Thus, the only poles inside C are z =0 and z = α.

$$\text{Res}(z=\alpha)=\lim_{z\to\alpha}\frac{(z-\alpha)(z^2-1)^4}{z^4(z-\alpha)(z-\beta)}=\frac{(\alpha^2-1)^4}{\alpha^4(\alpha-\beta)}=\frac{\alpha^4(\alpha-\alpha^{-1})^4}{\alpha^4(\alpha-\beta)}$$

$$=(\alpha-\beta)^3=\frac{8(a^2-b^2)^{\frac{3}{2}}}{b^3}.$$

For finding the residue at $z = 0$ we have to obtain the coefficient of $\frac{1}{z}$ in the power series expansion of

$$\frac{(z^2-1)^4}{z^4\left(z^2+\frac{2az}{b}+1\right)}=\frac{1}{z^4}(1-z^2)^4\left[1+\left(\frac{2az}{b}+z^2\right)\right]^{-1}$$

$$=\frac{1}{z^4}\left[1-4z^2+6z^4-4z^6+z^8\right]\left[1-\frac{2az}{b}-z^2+\frac{4a^2z^2}{b^2}+z^4+\frac{4az^3}{b}-\frac{8a^3z^3}{b^3}+...\right]$$

Coefficient of $\frac{1}{z}$ = Coefficient of z^3

$$=\frac{4a}{b}+\frac{8a}{b}-\frac{8a^3}{b^3}=\frac{12a}{b}-\frac{8a^3}{b^3}$$

So, by Cauchy residue theorem, we have

$$\mathrm{I}=\frac{2\pi i}{16ib}\left[\frac{8(a^2-b^2)^{\frac{3}{2}}}{b^3}+\frac{12ab^2-8a^3}{b^3}\right]=\frac{\pi}{b^4}\left[(a^2-b^2)^{\frac{3}{2}}-a^3+\frac{3}{2}ab^2\right].$$

4.2.8 Exercise:

(i) Prove that $\int_0^{\pi}\frac{a}{a^2+cos^2\theta}d\theta=\frac{\pi}{\sqrt{(a^2+1)}}$

(ii) Prove that $\int_0^{2\pi}e^{-\cos\theta}\cos(\sin\theta+\mathrm{n}\,\theta)\,\mathrm{d}\,\theta=\frac{2\pi(-1)^n}{n!}$, where n is a +ve integer.

(iii) Prove that $\int_0^{2\pi}\frac{1}{(2+\cos\theta)^2}d\theta=\frac{4\pi}{3\sqrt{3}}$

4.2.9 Theorem: Let *f(z)* be a function of the complex variable z satisfying the conditions

(i) *f(z)* is meromorphic in the upper half of the complex plane i.e. *Im(z)≥0.*

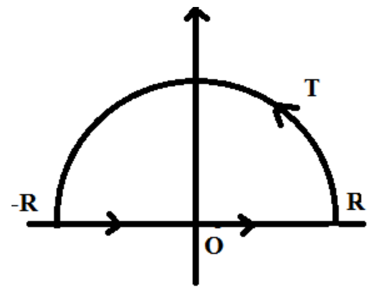
(ii) $f(z)$ has no pole on the real axis.

(iii) $zf(z)\to 0 \; uniformely \; as \; |z|\to\infty \; for \; o\le\arg z\le\pi.$

(iv) $\int_0^{\infty}f(x)dx \; and \int_{-\infty}^{\infty}f(x)dx$ both converge.

Then, $\int_{-\infty}^{\infty}f(x)dx=2\pi i\sum\mathrm{Re}\,s$ where $\sum\mathrm{Re}\,s.$ denotes the sum of residues of *f*(*z*) at its poles in the upper half of the z-plane.

Proof: Let us consider the integral $\int_C f(z)dz$, where C is the contour consisting of the segment of the real axis from –R to R and the semi-circle in the upper half plane on it as diameter.



Let the semi-circular part of the contour C be denoted by T and let R be chosen so large that C includes all the poles of $f(z)$.

Then by Cauchy's residue theorem,

$$\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_T f(z)dz = 2\pi i \sum res. \qquad (1)$$

By hypothesis (iii), $|zf(z)| < \in$ for all points z on T. If R is chosen sufficiently large, however small be the positive number $\in$, then for such R,

$$\left|\int_T f(\text{z})\,\text{dz}\right| = \left|\int_0^{\pi} f(\text{Re}^{i\theta})\,\text{Re}^{i\theta}\, id\theta\right|$$

$$= \left|\int_0^{\pi} zf(\text{z})\,\text{d}\,\theta\right| < \in \int_0^{\pi} d\theta = \in \pi\,.$$

It follows that as $|z| = R \to \infty, \int_T f(z)dz \to 0$.

Now, since hypothesis (iv) holds

$$\therefore \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx$$

Taking limit as $\text{R} \to \infty$ in (1), we get

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum \text{Re}\, s.$$

4.2.10 Example: By method of contour integration, prove that

$$\int_0^{\infty} \frac{dx}{(1+x^2)^2} = \pi/4$$

Solution: Consider the integral

$$\int_C f(z)dz, \text{ where } f(z) = \frac{1}{(1+z^2)^2}$$

C being the closed contour consisting of T, the upper half of the large circle $|z|$ =R and the real axis from –R to R. Poles of $f(z)$ are $z = \pm i$ (each of order two) $f(z)$ has only one pole of order two at z = i within C. We can write

$$f(z) = \frac{\phi(z)}{(z-i)^2}, \text{ where } \phi(z) = \frac{1}{(z+i)^2}$$

$$Re\,s(z=i) = \frac{1}{1!}\phi'(i) = \frac{1}{4i}$$

Hence, by Cauchy's residue theorem

$$\int_C f(z)dz = 2\pi i \frac{1}{4i} = \frac{\pi}{2}$$

`or $\int_T f(z)dz + \int_{-R}^{R} f(x)\,dx = \frac{\pi}{2}$

i.e. $\int_T \frac{dz}{(z^2+1)^2} + \int_{-R}^{R} \frac{dx}{(1+x^2)^2} = \frac{\pi}{2}$

Now, using the inequality,

$$|z_1 + z_2| \ge |z_1| - |z_2|, \frac{1}{|z_1 + z_2|} \le \frac{1}{|z_1| - |z_2|}, \text{ we get}$$

$$\left|\int_T \frac{dz}{(1+z^2)^2}\right| \le \int_T \frac{|dz|}{(z^2-1)^2} = \int_T \frac{|dz|}{(|z|^2-1)^2}$$

$$= \frac{1}{(R^2-1)^2}\int_T |dz| = \frac{\pi R}{(R^2-1)^2} \to 0 \text{ as } |z| = R \to \infty$$

So $\lim_{R\to\infty} \int_T \frac{dz}{(1+z^2)^2} = 0$

Making $R \to \infty$ in (1), we obtain

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \pi/2 \quad \text{or} \quad \int_0^{\infty} \frac{dx}{(1+x^2)^2} = \pi/4$$

4.2.11 Example: Prove that $\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9}.dx = \frac{5\pi}{12}$

Solution: Consider the integral $\int_C f(z)dz$ where $f(z) = \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}$, C be the closed contour consisting of T, the upper half of the large circle $|z| = R$ and the real axis from –R to R.

We know that $|z_1 \pm z_2| \ge |z_1| - |z_2|$

$$\Rightarrow \frac{1}{|z_1 \pm z_2|} \le \frac{1}{|z_1| - |z_2|}$$

Under this inequality, we have

$$\left|\int_T f(z)dz\right| = \left|\int_T \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}.dz\right|$$

$$\le \int_T \frac{|z|^2 + |z| + 2}{|z|^4 - 10|z|^2 - 9}.|dz| = \int_0^{\pi} \frac{R^2 + R + 2}{R^4 - 10R^2 - 9}.R.d\theta$$

$$= \frac{R^2 + R + 2}{R^4 - 10R^2 - 9}.\pi R \to 0 \; as\, R \to \infty.$$

Therefore, $\int_T f(z)dz = 0 \; as\, R \to \infty$.

Now, pole of *f(z)* are given by $z^4+10z^2+9=0 \Rightarrow (z^2+1)(z^2+9)=0 \quad \Rightarrow z=\pm i, z=\pm 3i$.
Out of these pole only two simple pole z = i and z= 3i lies within c.

Therefore, the sum of residues, Res(z=i)+Res(z=3i)=$\dfrac{5}{24i}$.

So, by Cauchy residue theorem, we get

$$\int_C f(z)dz = \int_T f(z)dz + \int_{-R}^{R} f(x)dx = 2\pi i\left(\frac{5}{24i}\right) = \frac{5\pi}{12}.$$

Making $R\to\infty$ and using (1), we get

$$\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}.dx = \frac{5\pi}{12}$$

4.2.12 Example: Prove that $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)^2} = \frac{\pi(a+2b)}{2ab^3(a+b)^2}$, $a>0, b>0$

Solution: Consider the integral $\int_C f(z)dz,$

where $f(z) = \dfrac{1}{(z^2+a^2)(z^2+b^2)^2}$

and C is the closed contour consisting of T, the upper half of the large circle $|z|=R$ and the real axis from –R to R. Poles of $f(z)$ are $z=\pm ia$ (simple) and $z=\pm ib$ (double). Only poles of $f(z)$ lying within C are$z=ia$ (simple) and $z=ib$ (double)

$$Res(z=ia) = \frac{1}{2ia(a^2-b^2)^2}$$

$$Res(z=ib) = \frac{(3b^2-a^2)i}{4b^3\left(a^2-b^2\right)^2}$$

Thus, sum of residues$= \dfrac{i}{4(a^2-b^2)^2}\left[\dfrac{-2}{a} + \dfrac{3b^2-a^2}{b^3}\right]$

$$= \frac{i\left[-2b^3+a(3b^2-a^2)\right]}{4ab^3(a^2-b^2)^2}$$

$$= \frac{i\left[(b^3-a^3)-3b^2(b-a)\right]}{4aba^3(a^2-b^2)^2}$$

$$= \frac{i\left[(b^3-a^3)-3b^2(b-a)\right]}{4ab^3(a^2-b^2)^2}$$

$$= \frac{i(b-a)\left[b^2+a^2+ab-3b^2\right]}{4ab^3(a^2-b^2)^2}$$

$$= \frac{i(b-a)(a-b)(a+2b)}{4ab^3(a^2-b^2)^2} = \frac{-i(a+2b)}{4ab^3(a+b)^2}$$

So, by Cauchy residue theorem,

$$\int_C f(z)\,dz = \int_T f(z) + \int_{-R}^{R} f(x)\,dx$$

$$= 2\pi i\left[\frac{-i(a+2b)}{4ab^3(a+b)^2}\right] = \frac{\pi(a+2b)}{2ab^3(a+b)^2}$$

Now, $$\left|\int_T f(z)\,dz\right| = \left|\int_T \frac{dz}{(z^2+a^2)(z^2+b^2)^2}\right|$$

$$= \left|\int_T \frac{|dz|}{(|z|^2-a^2)(|z|^2-b^2)^2}\right|$$

$$= \frac{\pi R}{(R^2-a^2)(R^2-b^2)} \to 0 \; as\, |z| = R \to \infty$$

Making $R \to \infty$ in (1), we obtain

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+z^2)(x^2+b^2)^2} = \frac{\pi(a+2b)}{2ab^3(a+b)^2}$$

4.2.13 Deductions:

(i) $$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)^2} = \frac{5\pi}{144}$$

(ii) $$\int_{-\infty}^{\infty} \frac{dx}{(x^2+2)(x^2+1)^2} = \frac{\pi}{9}$$

(iii) $$\int_{-\infty}^{\infty} \frac{dx}{(x^2+9)(x^2+4)^2} = \frac{7\pi}{1200}$$

4.2.14 Jordan Inequality: If $0 \le \theta \le \frac{\pi}{2}$, then$\frac{2\theta}{\pi} \le sin\theta \le \theta$. This inquality is called Jordan inequality.

We know that as θ increases from *0* to $\frac{\pi}{2}$, *cosθ* decreases steadily and consequently the mean ordinate of the graph of $y = cosx$ over the range $0 \le x \le \theta$ also decreases steadily and mean ordinate is given by

$$\frac{1}{\theta}\int_o^{\theta} \cos x dx = \frac{\sin\theta}{\theta}$$

It follows that when $0 \le \theta \le \frac{\pi}{2}$, we get $\frac{2}{\pi} \le \frac{\sin\theta}{\theta} \le 1$ or $\frac{2\theta}{\pi} \le \sin\theta \le \theta$.

4.2.15 Jordan Lemma: If *f(z)* is analytic except at a finite number of singularities and if $f(\mathrm{z}) \to 0$ uniformly as $z \to \infty$, then

$$\lim_{R\to\infty}\int_T e^{imz} f(\mathrm{z})\,\mathrm{dz} = 0,\; m > 0.$$

Where T denotes the semicircle $|z| = R$ and $\text{Im}(z) \geq 0$, R being taken so large that all the singularities of $f(z)$ lie within T.

Proof: Since $f(z) \to 0$ uniformly as $|z| \to \infty$ there exist $\in > 0$ such that $|f(z)| < \in \ \forall \ z$ on T.

Also, $\quad |z| = R \Rightarrow z = \text{Re}^{i\theta}$

$$dz = \text{Re}^{i\theta}\, id\theta \ \ and \ \ |dz| = Rd\theta$$

$$\left|e^{imz}\right| = \left|e^{im\text{Re}^{i\theta}}\right| = \left|e^{imR\cos\theta}.e^{-mR\sin\theta}\right| = e^{-mR\sin\theta}$$

Hence, using Jordan inequality, we get

$$\left|\int_T e^{imz} f(z)\,\text{dz}\right| \leq \int_T \left|e^{imz}\right|\left|f(z)\right|\left|dz\right|$$

$$< \int_0^{\pi} e^{-mR\sin\theta} \in Rd\theta$$

$$= 2 \in R\int_0^{\frac{\pi}{2}} e^{-mR\sin\theta}\, d\theta \begin{cases} \frac{2\theta}{\pi} \leq sin\theta \leq \theta \\ \frac{-2\theta}{\pi} \geq -sin\theta \end{cases}$$

$$\leq 2 \in R\int_0^{\frac{\pi}{2}} e^{\frac{-2mR\theta}{\pi}}\, d\theta$$

$$= 2 \in R\left[\frac{e^{\frac{-2mR\theta}{\pi}}}{\frac{-2mR}{\pi}}\right]_0^{\frac{\pi}{2}} = 2 \in R\left[e^{-mR} - e^{0}\right]\left(\frac{-\pi}{2mR}\right)$$

$$= \frac{-\pi \in}{m}\left(e^{-mR} - 1\right) = \frac{\in \pi}{m}\left(1 - e^{-mR}\right) < \frac{\in \pi}{m}.$$

So, $\lim_{R\to\infty} \int_T e^{imz} f(z)\,\text{dz} = 0$.

4.2.16 Integrals of the form $\int_{-\infty}^{\infty} f(x)\cos\alpha x\, dx$ and $\int_{-\infty}^{\infty} f(x)\sin\alpha x\, dx$: In view of Euler's formula, $e^{i\alpha x} = \cos\alpha x + i\sin\alpha x$, where α is a positive real number, we can write

$$\int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx = \int_{-\infty}^{\infty} f(x)\cos\alpha x\, dx + i\int_{-\infty}^{\infty} f(x)\sin\alpha x\, dx\ ,$$

whenever both integrals on the right hand side converge.

4.2.17 Example: By the method of contour integration, prove that

$$\int_0^{\infty} \frac{\cos mx}{x^2 + a^2} dx = \frac{\pi}{2a} e^{-ma} \ \text{, where } m \geq 0,\ a \geq 0.$$

Solution: Consider the integral $\int_C f(\mathrm{z})\,\mathrm{dz}$, where $f(\mathrm{z})=\dfrac{e^{imz}}{z^2+a^2}$ and C being a closed contour consisting of T, the upper half of the large circle $|z|=R$ and the real axis from –R to R.

Now, $\dfrac{1}{z^2+a^2}\to 0$ as $|z|=R\to\infty$.

Hence, by Jordan Lemma

$$\lim_{R\to\infty}\int_T \frac{e^{imz}}{z^2+a^2}\,dz=0$$

$$\text{or } \lim_{R\to\infty}\int_T f(\mathrm{z})\,\mathrm{dz}=0,\ \text{where } f(\mathrm{z})=\frac{e^{imz}}{z^2+a^2}\ . \qquad (1)$$

Now poles of $f(z)$ are given by $z^2+a^2=0$ i.e. $z=\pm ia$. But $z=ia$ is the only simple pole lying inside C.

$$\mathrm{Res}(z=ia)=\lim_{z\to ia}\frac{(\mathrm{z}-\mathrm{ia})\,\mathrm{e}^{imz}}{(\mathrm{z}+\mathrm{ia})(\mathrm{z}-\mathrm{ia})}$$

$$=\lim_{z\to ia}\frac{e^{imz}}{z+ia}=\frac{e^{-ma}}{2ia}.$$

Hence, by Cauchy residue theorem

$$\int_C f(\mathrm{z})\,\mathrm{dz}=\int_T f(\mathrm{z})\,\mathrm{d}z+\int_{-R}^{R} f(\mathrm{x})\,\mathrm{dx}$$

But $\displaystyle\int_C f(\mathrm{z})\,\mathrm{dz}=2\pi i\frac{e^{-ma}}{2ia}=\frac{\pi}{a}e^{-ma}$

$$\int_T f(\mathrm{z})\,\mathrm{dz}+\int_{-R}^{R} f(\mathrm{x})\,\mathrm{dx}=\frac{\pi}{a}e^{-ma}$$

Making $R\to\infty$ and using (1), we obtain

$$\int_{-\infty}^{\infty}\frac{e^{imx}}{x^2+a^2}dx=\frac{\pi}{a}e^{-ma}$$

Equating the real parts on both sides, we have

$$\int_{-\infty}^{\infty}\frac{\cos mx}{x^2+a^2}dx=\frac{\pi}{a}e^{-ma}\qquad \text{or } \int_0^{\infty}\frac{\cos mx}{x^2+a^2}dx=\frac{\pi}{2a}e^{-ma}$$

4.2.18 Deductions:

(i) $\displaystyle\int_0^{\infty}\frac{\cos ax}{x^2+1}dx=\frac{\pi}{2}e^{-a}$

(ii) $\displaystyle\int_0^{\infty}\frac{\cos x}{x^2+1}dx=\frac{\pi}{2}e^{-1}=\frac{\pi}{2e}$

(iii) $\displaystyle\int_0^{\infty}\frac{\cos x}{x^2+4}dx=\frac{\pi}{4e^2}$

4.2.19 Example: Prove that $\int_0^\infty \frac{x\sin x}{x^2+a^2} = \frac{\pi}{2}e^{-a}, a>0$

Solution: Consider the integral $\int_C f(z)dz$ where $f(z)=\frac{ze^{iz}}{z^2+a^2}$ and C being a closed contour consisting of T, the upper half of the large circle $|z|=R$ and the real axis from –R to R.

Now $\frac{z}{z^2+a^2}\to 0$ as $|z|\to R\to\infty$. Therefore, Res $(z=ia)$ $=\frac{e^{-a}}{2}$.

Hence, by Jordan's lemma

$$\lim_{R\to\infty}\int_T f(z)dz = 0$$

Now, by Cauchy residue theorem

$$\int_C f(z)dz = \int_T f(z)dz + \int_{-R}^{R} f(x)dx = \frac{2\pi i e^{-a}}{2}$$

Making $R\to\infty$ and using (1), we obtain

$$\int_{-\infty}^{\infty}\frac{xe^{ix}}{x^2+a^2}.dx = \pi i e^{-a}$$

$$\therefore \quad \int_0^\infty \frac{xe^{ix}}{x^2+a^2}.dx = \frac{\pi i e^{-a}}{2}$$

Equating imaginary parts, we obtain

$$\int_0^\infty \frac{x\sin x}{x^2+a^2}.dx = \frac{\pi}{2}e^{-a}$$

Similarly, $\int\limits_0^\infty \frac{x\sin x}{x^2+9}dx \;= \frac{\pi}{2}e^{-3}$

4.2.20 Example: Prove that $\int_0^\infty \frac{x^3\sin mx}{x^4+a^4}.dx = \frac{\pi}{2}e^{\frac{-ma}{\sqrt{2}}}\cos\frac{ma}{\sqrt{2}}, a>0, m>0$

Solution: Consider the integral $\int_C f(z)dz$ where $f(z)=\frac{z^3e^{imz}}{z^4+a^4}$ and C is the closed contour consisting of the upper half of the large circle and real axis from –R to R.

Since $\frac{z^3}{z^4+a^4}\to 0$ as $|z|\to R\to\infty$ .So by Jordan lemma, we get

$$\lim_{R\to\infty}\int_T \frac{z^3e^{imz}}{z^4+a^4}dz = 0$$

$$\text{i.e. } \lim_{R\to\infty}\int_T f(z)dz = 0$$

Now poles of *f(z)* are given by $z^4+a^4=0$ i.e $z^4=-a^4$

$$\therefore z^4 = e^{2n\pi i}.e^{\pi i}a^4 = e^{(2n+1)\pi i}a^4$$

$$\therefore z = e^{\frac{(2n+1)\pi i}{4}a}$$, where n =0,1,2,3…

Out of these four poles only $z = ae^{\frac{i\pi}{4}}$ and $z = ae^{\frac{i3\pi}{4}}$ lie inside C.

If $f(z) = \dfrac{\varphi(z)}{\psi(z)}$ thenRes(z=α)=$\lim_{z\to\alpha}\dfrac{\varphi(z)}{\psi'(z)}$, α being simple poles.

For the present case, we have

$$\text{Res}(z=\alpha) = \lim_{z\to\alpha}\frac{z^3 e^{imz}}{4z^3} = \lim_{z\to\alpha}\frac{e^{imz}}{4}$$

Therefore, sum of residues i.e. Res($z = ae^{\frac{i\pi}{4}}$) + Res($z = ae^{\frac{3i\pi}{4}}$)

$$= \frac{1}{4}\left[\exp\left(imae^{\frac{i\pi}{4}}\right) + \exp\left(imae^{\frac{3i\pi}{4}}\right)\right]$$

$$= \frac{1}{4}\left[\exp\left\{ima\left(\frac{1+i}{\sqrt{2}}\right)\right\} + \exp\left\{ima\left(\frac{-1+i}{\sqrt{2}}\right)\right\}\right]$$

$$= \frac{1}{4}\left[\exp\left(\frac{-ma}{\sqrt{2}}\right)\left\{\exp\left(\frac{ima}{\sqrt{2}}\right) + \exp\left(\frac{-ima}{\sqrt{2}}\right)\right\}\right]$$

$$= \frac{1}{2}\exp\left(\frac{-ma}{\sqrt{2}}\right)\cos\frac{ma}{\sqrt{2}}.$$

Hence, by Cauchy residue theorem, we get

$$\int_C f(z)\,dz = \pi i \exp\left(\frac{-ma}{\sqrt{2}}\right)\cos\frac{ma}{\sqrt{2}}$$

Taking limit $|z| = R \to \infty$ and using (1), we get

$$\int_{-\infty}^{\infty} f(x)\,dx = \pi i \exp\left(\frac{-ma}{\sqrt{2}}\right)\cos\left(\frac{ma}{\sqrt{2}}\right)$$

$$\int_{-\infty}^{\infty} \frac{x^3 e^{imx}}{x^4 + a^4}\,dx = \pi i \exp\left(\frac{-ma}{\sqrt{2}}\right)\cos\left(\frac{ma}{\sqrt{2}}\right)$$

Equating imaginary parts, we obtain

$$\int_{-\infty}^{\infty} \frac{x^3 \sin mx}{x^4 + a^4}\,dx = \pi \exp\left(\frac{-ma}{\sqrt{2}}\right)\cos\left(\frac{ma}{\sqrt{2}}\right)$$

$$or \int_0^{\infty} \frac{x^3 \sin mx}{x^4 + a^4}\,dx = \frac{\pi}{2}\exp\left(\frac{-ma}{\sqrt{2}}\right)\cos\left(\frac{ma}{\sqrt{2}}\right)$$

4.3 Conformal Mappings:

4.3.1 Definition: A mapping from z-plane to w-plane is called **isogonal** if it has a one-one transformation which maps any two intersecting curves of z-plane into two curves of w-plane which cut at the same angle. Thus, in an isogonal mapping, only the magnitude of angle is preserved.

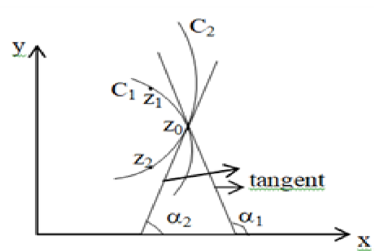
4.3.2 Definition: Let $w = f(z)$ be a one to one mapping from the z-plane into w-plane. Let C_1 and C_2 be two continuous arcs in z-plane through the point $z_0 = (x_0, y_0)$ which are mapped respectively into curves C_1' and C_2' intersecting at the point $w_0 = f(z_0)$. Then, if the angle at the point z_0 between the arcs C_1 and C_2 is equal to the angle at the point w_0 between C_1' and C_2' both in the magnitude and sense of rotation, then the mapping is called **conformal** mapping at point z_0.

4.3.3 Remarks: Some authors do not distinguish between isogonal and conformal mappings. They regard conformality as the preservation of the magnitude of angle without considering the sense of rotation.

The following theorem provides the necessary and sufficient condition for conformality which briefly states that if $f(z)$ is analytic, mapping is conformal.

4.3.4 Theorem: Prove that at each point z of a domain where $f(z)$ is analytic and $f'(z_0)$ is not zero, z_0 being an interior point, the mapping $w = f(z)$ is conformal.

Proof: Let $w = f(z)$ be an analytic function of z, regular and one valued in a domain D of the z-plane. Let z_0 be an interior point of D. Let C_1 and C_2 be continuous curves passing through the point z_0 and having definite tangents at this point making angles α_1 and α_2, say, with real axis.

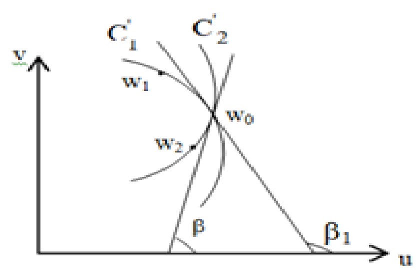


We have to discover that what is the representation of this figure in the w-plane. Let z_1 and z_2 be two points on the curves C_1 and C_2 respectively, where z_1 and z_2 are taken very close to z_0. We shall suppose that they are at the same distance r from z_0 so that we can write

$$|z_1 - z_0| = r \text{ and } |z_2 - z_0| = r$$

i.e. $z_1 - z_0 = re^{i\theta_1}$ and $z_2 - z_0 = re^{i\theta_2}$.

Then, as $r \to 0$, $\theta_1 \to \alpha_1$ and $\theta_2 \to \alpha_2$.



The point z_0 corresponds to a point w_0 in the w-plane and the points z_1 and z_2 corresponds to points w_1 and w_2 which describe curves C_1' and C_2' making angles β_1 and β_2 with the real axis as shown in the figure. Let

$$w_1 - w_0 = \rho_1 e^{i\phi_1}$$
$$w_2 - w_0 = \rho_2 e^{i\phi_2}$$

when ρ_1 and $\rho_2 \to 0$, then $\phi_1 \to \beta_1$ and $\phi_2 \to \beta_2$ respectively.

Now, by definition of analytic function, as $f(z)$ is given to be analytic, we have

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z_1 \to z_0} \frac{w_1 - w_0}{z_1 - z_0}$$
$$= \lim_{z_1 \to z_0} \frac{\rho_1 \, e^{i\phi_1}}{r \, e^{i\theta_1}} = R\, e^{i\delta}$$

i.e., $\displaystyle\lim_{z_1 \to z_0} \frac{\rho_1}{r} e^{i(\varphi_1 - \theta_1)} = \mathrm{Re}^{i\delta}$

Equating the argument and modulus, we have

$$\lim_{z_1 \to z_0} \frac{\rho_1}{r} = R = |f'(z_0)| \text{ and}$$

$$\lim_{z_1 \to z_0} (\varphi_1 - \theta_1) = \delta$$

i.e, $\beta_1 - \alpha_1 = \delta \;\Rightarrow\; \beta_1 = \alpha_1 + \delta$

Similarly, we have $\beta_2 = \alpha_2 + \delta$.

Hence, the curves C_1' and C_2' have definite tangents at point w_0 making angle $\alpha_1 + \delta$ and $\alpha_2 + \delta$ with the real axis. Thus, the angle between C_1' and C_2' is

$$\beta_1 - \beta_2 = (\alpha_1 + \delta) - (\alpha_2 + \delta) = \alpha_1 - \alpha_2 \; .$$

Which is the same as the angle between C_1 and C_2. Hence, the curves C_1' and C_2' intersect at the same angle as the curves C_1 and C_2. Also, the angle between two curves has the same sense in the two figures and therefore the mapping is the conformal mapping.

4.3.5 Special Case: In the above theorem, when $f'(z) = 0$.

Proof: Suppose $f'(z)$ has a zero of order n at the point z_0, so that

$$f'(z) = (z - z_0)^n \varphi(z) \; ,$$

where $\varphi(z)$ is analytic and $\varphi(z_0) \neq 0$

$$\Rightarrow \quad f'(z_0) = f''(z_0) = f'''(z_0) = \dots = f^n(z_0), \;\; f^{n+1}(z_0) \neq 0 . \qquad (1)$$

Expanding $f(z)$ by Taylor's theorem in the neighborhood of z_0 , we have

$$f(z) = \sum_{m=0}^{\infty} a_m (z - z_0)^m \qquad , \qquad (2)$$

where $\quad a_m = \dfrac{f^m(z_0)}{m!} \qquad (3)$

Applying (1) to (3), we have $a_m = 0$ for $m = 1, 2, 3, ..., n$.

Thus, from (2), we have

$$f(z) = a_0 (z - z_0)^0 + \sum_{m=n+1}^{\infty} a_m (z - z_0)^m \ .$$

But $\quad a_0 = \dfrac{f^{(0)}(z_0)}{0!} = f(z_0)$

$$\therefore \quad f(z) - f(z_0) = a_{n+1}(z_1 - z_0)^{n+1} + ...$$

$$\Rightarrow \qquad \mathrm{w}_1 - w_0 = a_{n+1}(z_1 - z_0)^{n+1}$$

Taking, $w_1 - w_0 = \rho_1 e^{i\varphi_1}$, $z_1 - z_0 = re^{i\theta_1}$ and $a_{n+1} = a\, e^{i\lambda}$.

Therefore, we get

$$\rho_1\, e^{i\varphi_1} = a\, e^{i\lambda}\, r^{n+1}\, e^{i(\mathrm{n}+1)\theta_1}$$
$$= a\, r^{n+1}\, e^{i[(n+1)\theta_1 + \lambda]}$$

$$\Rightarrow \ \lim \varphi_1 = \lim\{(\mathrm{n}+1)\theta_1 + \lambda\} = (\mathrm{n}+1)\alpha_1 + \lambda$$

Similarly, $\quad \lim \varphi_2 = (\mathrm{n}+1)\alpha_2 + \lambda$

$$\Rightarrow \ \lim(\phi_2 - \phi_1) = (n+1)(\alpha_2 - \alpha_1)$$

The curves C_1' and C_2' have definite tangents at w_0 but the angle between them is $n+1$ times the angle between C_1 and C_2 passing through z_0. Consequently, conformal property does not hold at z_0.

4.3.6 Definition: If a complex function $f(z)$ is analytic at a point z_0 and if $f'(z_0) = 0$, then z_0 is called a critical point of $f(z)$.

The following theorem is the converse of the above theorem and a sufficient condition for conformal mapping.

4.3.7 Theorem: If the mapping $w = f(z)$ is conformal and there exist a pair of continuously differentiable relations $u = u(x, y)$, $v = v(x, y)$ then show that $f(z)$ is an analytic function of z .

Proof: Let $u = u(x, y)$, $v = v(x, y)$ be a pair of differentiable equation defining conformal transformation from *xy*-plane to *uv*-plane. Let ds and $d\sigma$ be the length elements in the *xy*-plane and *uv*-plane respectively, then by definition

$$\left.\begin{aligned} ds^2 &= dx^2 + dy^2 \\ d\sigma^2 &= du^2 + dv^2 \end{aligned}\right\} \qquad (1)$$

Since *u* and *v* are functions of *x* and *y*, therefore

$$du = \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy \text{ and } dv = \frac{\partial v}{\partial x}\, dx + \frac{\partial v}{\partial y}\, dy$$

$$\therefore\; du^2 + dv^2 = \left[\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right]^2 + \left[\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right]^2$$

i.e., $$d\sigma^2 = \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right](dx)^2 + \left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right](dy)^2 + 2\left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right]dx\,dy \qquad (2)$$

Since the mapping is conformal, therefore the ratio $d\sigma^2 : ds^2$ is independent of the direction. Comparing the coefficient from equation (1) and (2), we have

$$\frac{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}{1} = \frac{\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}}{0} = \frac{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2}{1}$$

$$\Rightarrow \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \qquad (3)$$

and $$\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = 0 \qquad (4)$$

From equation (4), we have

$$\frac{\partial u / \partial x}{\partial v / \partial y} = \frac{\partial v / \partial x}{i\partial u / \partial y} = \lambda(\text{say})$$

$$\therefore\; \frac{\partial u}{\partial x} = \lambda\frac{\partial v}{\partial y},\quad \frac{\partial v}{\partial x} = -\lambda\frac{\partial u}{\partial y} \qquad (5)$$

Putting this in equation (3), we get

$$\lambda^2\left(\frac{\partial v}{\partial y}\right)^2 + \lambda^2\left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2$$

or $$(\lambda^2 - 1)\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] = 0$$

$$\Rightarrow\; \lambda^2 - 1 = 0 \;\Rightarrow\; \lambda = \pm 1$$

Using this in equation (5), we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad (\text{when } \lambda=1) \qquad (6)$$

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y},\quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \qquad (\text{when } \lambda= -1) \qquad (7)$$

The equation (6) is Cauchy- Riemann equation and hence $w = f(z)$ is an analytic function. The equation (7) are reduced to (6) by writing $-v$ in place of v i.e., by taking as image figure obtained by the reflection in the real axis of the w-plane. Thus, the four partial derivatives u_x, u_y, v_x, v_y exist are continuous and satisfy C-R equations. Hence, $f(z)$ is analytic.

4.3.8 Remarks:

(i) The mapping $w = f(z)$ is conformal in a domain D if it is conformal at each point of the domain.

(ii) The conformal mappings play an important role in the study of various physical phenomena defined on domains and curves of arbitrary shapes. Smaller portions of these domains and curves are conformally mapped by analytic function to well-known domains and curves.

4.3.9 Example: Find all points where the mapping $f(z) = \sin z$ is conformal.

Solution: The function $f(z) = \sin z$ is entire and $f'(z) = \cos z = 0$ if and only if $z = (2n+1)\dfrac{\pi}{2},\ n \in 0, \pm 1, \pm 2....$ and so each of these points is a critical point of f. By theorem 4.3.4, $\sin z$ is a conformal mapping for all $z \neq (2n+1)\dfrac{\pi}{2},\ n \in 0, \pm 1, \pm 2.....$ Furthermore, $w = \sin z$ is not a conformal mapping at $z = (2n+1)\dfrac{\pi}{2},\ n \in 0, \pm 1, \pm 2....$ because $f''(z) = -\sin z = \mp 1$ atthe critical points of f.

4.3.10 Example: Discuss the mapping $w = \bar{z}$.

Solution: We observe that the given mapping replaces every point by its reflection in the real axis. Hence, angles are conserved but their signs are changed and thus the mapping is isogonal but not conformal. If the mapping $w = \bar{z}$ is followed by a conformal transformation, then resulting transformation of the form $w = f(\bar{z})$ is also isogonal but not conformal, where $f(z)$ is analytic function of z.

4.3.11 Example: Discuss the mapping $w = z^a$, where a is a positive real number.

Solution: Denoting z and w in polar as

$$z = re^{i\theta}, w = \rho e^{i\phi}, \text{ the mapping gives } \rho = r^a,\ \phi = a\theta.$$

Thus the radii vectors are raised to the power a and the angles with vertices at the origin are multiplied by the factor a. If $a > 1$, distinct lines through the origin in the z-plane are not mapped onto distinct lines through the origin in the w-plane, since, e.g. the straight line through the origin at an angle $\dfrac{2\pi}{a}$ to the real axis of the z-plane is mapped onto a line through the origin in the w-plane at an angle 2π to the real axis i.e. the positive real axis itself. Further $\dfrac{dw}{dz} = az^{a-1}$, which vanishes at the origin if a > 1 and has a singularity at the origin if a < 1. Hence the mapping is conformal and the angles are therefore preserved, excepting at the origin.

Similarly, the mapping $w = e^z$ is conformal.

4.3.12 Example: Prove that the quadrant $|z|<1$, $0<\arg z<\dfrac{\pi}{2}$ is mapped conformally onto a domain in the w-plane by the transformation $w=\dfrac{4}{(z+1)^2}$.

Solution: If $w=f(z)=\dfrac{4}{(z+1)^2}$, then $f'(z)$ is finite and does not vanish in the given quadrant. Hence, the mapping $w=f(z)$ is conformal and the quadrant is mapped onto a domain in the w-plane provided w does not assume any value twice i.e. distinct points of the quadrant are mapped to distinct points of the w-plane. We show that this indeed is true. If possible, let $\dfrac{4}{(z_1+1)^2}=\dfrac{4}{(z_2+1)^2}$, where $z_1 \neq z_2$ and both z_1 and z_2 belong to the quadrant in the z-plane. Then, since $z_1 \neq z_2$, we have $(z_1-z_2)(z_1+z_2+2)=0$.

$\Rightarrow (z_1+z_2+2)=0$ i.e. $z_1=-z_2-2$. But since z_2 belongs to the quadrant, $-z_2-2$ does not, which contradicts the assumption that z_1 belongs to the quadrant. Hence, w does not assume any value twice.

4.3.13 Exercise: Determine where the complex mapping $w=f(z)$ is conformal.

1. $f(z)=z^3-3z+1$
2. $f(z)=z^2+2iz-3$
3. $f(z)=z-e^{-z}+1-i$
4. $f(z)=ze^{z^2-2}$
5. $f(z)=\tan z$
6. $f(z)=z-\ln(z+i)$

4.3.14 Exercise: Show that the complex mapping $w=f(z)$ is not conformal at the indicated point.

1. $f(z)=(z-i)^3;\ z_0=i$
2. $f(z)=(iz-3)^2;\ z_0=-3i$
3. $f(z)=e^{z^2};\ z_0=0$
4. $f(z)=\sqrt{z}\ ;\ z_0=0$

4.4 Space of Analytic Function:

4.4.1 Definition: A metric space is a pair (*X, d*) where *X* is a set and *d* is a function from $X \times X$ into R, called the distance function or metric, which satisfy the following conditions for $x, y, z \in X$

(i) $d(x, y) \geq 0$

(ii) $d(x, y) = 0$ *if* $x = y$

(iii) $d(x, y) = d(y,x)$

(iv) $d(x, z) \le d(x, y) + d(y,z)$

Conditions (iii) and (iv) are called 'symmetry' and 'triangle inequality' respectively. A metric space (*X, d*) is said to be bounded if there exists a positive number K such that $d(x, y) \le K$ for all $x, y \in X$. The metric space (*X, d*), in short, is also denoted by *X*, the metric being understood. If *x* and *r*> 0 are fixed then let us define

$$B(x;r) = \{x \in X : d(x,y) < r\}$$

$$\overline{B}(x;r) = \{y \in X : d(x,y) \le r\}$$

B(x; r) and $\overline{B}(x;r)$ are called open and closed balls (spheres) respectively, with centre*x* and radius *r*. $B(x;\varepsilon)$ is also referred to as the ε -neighbourhood of*x*.

Let *X* = *R* or ℂ and define $d(z, w) = |z - w|$. This makes both (*R, d*) and (ℂ , *d*) metric spaces. (ℂ , *d*)is the case of principal interest for us. In (ℂ , *d*), open and closed balls are termed as open and closed discs respectively. A metric space (*X, d*) is said to be **complete** if every sequence in *X* converges to a point of *X*, R and ℂ are examples of complete metric spaces. If G is an open set in ℂ and (X, d) is complete metric space then the set of all continuous functions from G to X is denoted by C(G, X).

The set C(G, X) is always non empty as it contains the constant functions. However it is possible that C(G, X) contains only the constant functions. For example, suppose that G is connected and X = N = {1, 2, 3, 4,..}. If $f \in$C(G, X) then *f(G)* must be connected in X and hence, must be singleton as the only connected subsets of N are singleton sets.

In this section we shall be mainly concern with the case when *X* is either ℂ or $\mathbb{C}_\infty$.To put a metric on C(G, X), we need the following results.

4.4.2 Theorem: If *G* is openset in ℂthen there is a sequence $\{K_n\}$ of compact subsets of G such that $G = \bigcup_{n=1}^{\infty} K_n$. Moreover, the sets $\{K_n\}$ can be chosen to satisfy the following conditions:

(i) $K_n \subset \operatorname{int} K_{n+1}$

(ii) $K \subset G$ and K compact implied $K \subset K_n$ for some n.

4.4.3 Definition: If G is open set inℂand $G = \bigcup_{n=1}^{\infty} K_n$ where each K_n is compact and $K_n \subset \operatorname{int} K_{n+1}$. For $n \in N$, we define

$\rho_{n(f,g)} = \sup\{d(f(z), g(z)) : z \in K_n\}$,for all functions *f* and *g* in C(G, X).

Also, if we define

$$\rho(f,g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f,g)}{1+\rho_n(f,g)}, \text{ for all } f,g \in C(G,X).$$

Then, $(C(G, X),\rho)$ is a **metric space**. In fact $(C(G, X),\rho)$ is a complete metric space.

4.4.4 Definition: A set $\Phi \subset C(G,X)$ is **normal** if each sequence Φ has a subsequence which converges to a function *f* in $C(G,X)$.

4.4.5Lemma:A set $\Phi \subset C(G,X)$ is normal iff its closure is compact.

4.4.6 Definition:A set $\Phi \subset C(G,X)$ is called **equicontinuous at a point *z₀*** in G iff for every $\varepsilon > 0$ there is $\delta > 0$ such that for $|z - z_0| < \delta$,

$$d\left(f(z), f(z_0)\right) < \varepsilon \text{, for every } f \text{ in } \Phi .$$

A set Φ is said to be **equicontinuous over a set** $E \subset G$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for z and z' in E and $|z - z'| < \delta$, we have

$$d\left(f(z), f(z')\right) < \varepsilon \text{, for all } f \text{ in } \Phi .$$

Notice that if Φ is consist of a single function *f* then the statement that Φ is equicontinuous at $z_0$is only the statement that *f* is continuous at z_0. The important thing about equicontinuous is that the same δ will work for all the functions in Φ. Also for $\Phi = \{f\}$ to be equicontinuous over E is equivalent to the uniform continuity of *f* on E.Further, suppose $\Phi \subset C(G,X)$ is equicontinuous at each point of G then Φ is equicontinuous over each compact subset of G.

4.4.7 Arzela-Ascoli Theorem: A set $\Phi \subset C(G,X)$ is normal iff the following two conditions are satisfied:

(i) For each z in G, $\{f(z); f \in \Phi\}$ has compact closure in X.

(ii) Φ is equicontinuous at each point of G.

Let G be an open subset of complex plane H(G) be the collection of holomorphic (analytic) functions on G.The following theorem shows that H(G) is a closed subset of C(G, ℂ).

4.4.8 Theorem: If $\{f_n\}$ is a sequence in H(G) and *f* belongs to C(G, ℂ) such that $f_n \to f$ then *f* is analytic and $f_n^{(k)} \to f^{(k)}$ for each integer $k \geq 1$.

Proof: To show f is analytic on G, we shall use the following form of Morera's theorem which states, "Let G be a region and let $f: G \to \mathbb{C}$ be a continuous function such that $\int_T f = 0$ for every triangular path T in G, then *f* is analytic in G". Let T be a triangle contained inside a disk $D \subset G$. Since T is compact, $\{f_n\}$ converges to *f* uniformly over T. Hence,

$$\int_T f = \lim \int_T f_n = 0 .$$

Since each f_n is analytic. Thus *f* must be analytic in every disk $D \subset G$. This gives that *f* is analytic in G. To show that $f_n^{(k)} \to f^{(k)}$, let D denote the closure of B(a, r) contained in G. Then there is a number R > r such that $\overline{B}(a;R) \subset G$.If γ is the circle $|z - a| = R$ then by Cauchy's integral formula,

$$f_n^{(k)}(z)-f^{(k)}(z)=\frac{k!}{2\pi i}\int_\gamma \frac{f_n(w)-f(w)}{(w-z)^{k+1}}dw, \text{ for z in D.}$$

Let $M_n = Sup\ \{|\ f_n(w)-f(w)|:|\ w-a|=R\}$. Then by Cauchy's estimate, we have

$$|\ f_n^{(k)}(z)-f^{(k)}(z)|\le \frac{k!M_n R}{(R-r)^{k+1}}, \text{ for } |\ z-a|\le r \qquad (1)$$

Since $f_n \to f$, lim M_n = 0.Thus, it follows from (1) that $f_n^{(k)} \to f^{(k)}$ uniformly on $\overline{B}(a;R)$. Now let K be an arbitrary compact subset of G and $0<r<d(K,\partial G)$ then there are $a_1,a_2,...a_n$ in K such that

$$K\subset \bigcup_{j=1}^{n} B(a_j;r)$$

Since $f_n^{(k)} \to f^{(k)}$ uniformly on each $B(a_j;r)$, it follows that $f_n^{(k)} \to f^{(k)}$ uniformly on K, which completes the proof of the theorem.

4.4.9 Corollary: H(G) is a complete metric space.

Proof: Since C(G, ℂ)is a complete metric space and H(G) is closed subset ofC(G, ℂ), we get that H(G) is also complete using "Let (X, d) be a complete metric space and $Y\subset X$. Then (Y, d) is complete iff Y is closed in X ".

4.4.10 Corollary: If $f_n : G \to \mathbb{C}$ is analytic and $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on compact sets to *f(z)* then $f^{(k)}z = \sum_{n=1}^{\infty} f_n^{(k)}(z)$.

4.4.11 Hurwitz's Theorem: Let G be a region and suppose the sequence $\{f_n\}$ in H(G) converges to *f*. If $f\ne 0$, $\overline{B}(a;R)\subset G$ and $f(z)\ne 0$ for $|\ z-a|=R$ then there is an integer N such that for $n\ge N$, *f* and f_n have the same number of zeros in B(a ;R).

Proof: Let $\delta=\inf\{|f(z)|:|z-a|=R\}$. Since $f(z)\ne 0$ for $|\ z-a|=R$, we have $\delta>0$.

Now $f_n \to f$ uniformly on {z : $|\ z-a|=R$ } so there is an integer N such that if $n \ \ge\ N$ and

$|\ z-a|=R$ then $|\ f(z)-f_n(z)|<\frac{\delta}{2}<|f(z)|$

Hence, by Rouche's theorem, *f* and f_n have the same number of zeros in B(a; R).

4.4.12 Corollary: If $\{f_n\}\subset H(G)$ converges to *f* in H(G) and each f_n never vanishes on G then either $f\equiv 0$ or *f* never vanishes.

4.4.13 Remark: Another form of **Hurwitz's theorem** is "Let $\{f_n(z)\}$ be a sequence of functions, each analytic in a region D bounded by a simple closed contour and let $f_n(z)\to f(z)$ uniformly in D. Suppose that f(z) is not identically zero. Let z_0 be an interior point of D. Then z_0 is a zero of *f(z)* if and only if it is a limit point of the set of zeros of the functions $f_n(z)$, points which are zeros of $f_n(z)$ for an infinity of values of n being counted as limit points."

Proof: Let z_0 be any point of D and let γ be a circle with centre z_0 and radius ρ so small that γ lies entirely in D. Suppose γ neither contains nor has on it any zero of *f(z)* except possibly for thepointz$_0$itself.Then,|*f*(z)|hasastrictlypositivelowerboundonthecircle $|z-z_0|=\rho$,say,

$$|f(z)| \geq K > 0 \qquad (1)$$

Having fixed ρ and K, we can choose N so large that, on the circle,

$$|f_n(z)-f(z)|<K \text{ for all } n>N \qquad (2)$$

From (1) and (2), we get

$$|f_n(z)-f(z)|< |f(z)|$$

Thus, if we set $g(z) = f_n(z)-f(z)$, then on the circle $|z-z_0|=\rho,\ |g(z)|<|f(z)|$.

Hence, by Rouche's theorem, for n > N, *g(z) + f(z)* i.e. f_n(z) has the same number of zeros as *f*(z) inside the circle γ . Thus, if *f*(z) = 0, then f_n(z) has exactly one zero inside γ for n > N. Therefore, z_0 is the limit point of the zeros of f_n(z). If $f(z_0)\neq 0$, then $f_n(z_0)\neq 0$ inside γ for n > N which completes the proof.

4.4.14 Definition : A set $\Phi \subset H(G)$ is called **locally bounded** if for each point a in G there are constants M and r > 0 such that for all f in Φ ,

$$|f(z)|\leq M \text{ for } |z-a| < r.$$

Alternately, Φ is locally bounded if there is an r > 0 such that

$$\sup\{|f(z)|:|z-a|<r, f\in\Phi\}<\infty.$$

That is, Φ is locally bounded if about each point a in G there is a disk on which Φ is uniformly bounded.

4.4.15 Lemma: A set Φ in H(G) is locally bounded iff for each compact set $K\subset G$ there is a constant M such that $|f(z)|\leq M$ **,** for all f in Φ and z in K.

4.4.16 Montel's Theorem: A family Φ in H(G) is normal iff Φ is locally bounded.

Proof: Suppose Φ is normal. We have to show Φ is locally bounded. Let, if possible, suppose that Φ is not locally bounded. Then there is a compact set $K\subset G$ such that

$$\sup\{|f(z)|: z\in K, f\in\Phi\}=\infty$$

That is, there is a sequence {f_n} in Φ such that

$$\sup\{| f_n(z)|: z\in K\}\geq n.$$

Since Φ is normal there is a function *f* in H(G) and a subsequence $\{f_{n_k}\}$ such that $f_{n_k}\to f$

This gives, $\lim_{k\to\infty} \sup\left(|f_{n_k}(z)-f(z)|: z\in K\right)=0$

Let $\left|f(z)\right|\le M$ for z in K. Then

$$\begin{aligned} n_k &\le \sup\{|f_{n_k}(z)|: z\in K\} \\ &= \sup\{\ |f_{n_k}(z)-f(z)\ +f(z)|: z\in\ K\} \\ &= \sup\{\ |f_{n_k}(z)-f(z)|: z\in K\}+\sup\{\ |f(z)|: z\in K\} \end{aligned}$$

$$\Rightarrow n_k \le \sup\{|f_{n_k}(z)-f(z)|: z\in K\}+M$$

$$\Rightarrow \lim_{k\to\infty} n_k \le M$$

A contradiction since $<n_k>$ is strictly monotonically increasing sequence. Hence our supposition is wrong. So Φ must be locally bounded.

Conversely, suppose that Φ is locally bounded. Then for each z in G, $\{f(z): f\in\Phi\}$ has compact closure. We now show that Φ is equicontinuous at each point of G. Let a be any fixed point of G and ε> 0. By hypothesis, there is an r > 0 and M > 0 such that $\overline{B}(a\ ;\ r)\subset G$ and $\left|f(z)\right|\le M$ for all z in $\overline{B}(a\ ;\ r)$ and for all f in Φ .

Let $\left|z-a\right|<\frac{1}{2}r$ and $f\in\Phi$. Then by Cauchy's formula, with $r(t)=a+re^{it}$, $0\ \le t\le 2\pi$,

$$|f(a)-f(z)|\le\frac{1}{2\pi}\left|\int_\gamma \frac{f(w)(a-z)}{(w-a)(w-z)}dw\right|$$

$$\le\frac{4M}{r}|a-z|$$

Choose δ straight line $0<\delta, \min\left\{\frac{r}{2},\frac{r}{4M}\varepsilon\right\}$.

Then $|a-z|<\delta$ gives $|f(a)\ -f(z)|<\varepsilon$ for all f in Φ . Therefore F is equicontinuous at $a\in G$. Hence, by Ascoli-Arzela theorem, Φ is normal.

4.4.17 Corollary: A set $\Phi\subset$ H(G) is compact iff it is closed and locally bounded.

4.4.18 Definition: A region G_1 is called **conformally equivalent** to G_2 if there is an analytic function f: $G_1\to\mathbb{C}$ such that f is one-one and $f(G_1)=G_2$.

It is immediate that $\mathbb{C}$ is not equivalent to any bounded region by Liouville's theorem. Also, it follows from the definition that if G_1 is simply connected and G_1 is equivalent to G_2, then G_2 must be simply connected.

We now prove Riemann mapping theorem which states that every simply connected region G in the plane (other than the plane itself) is conformally equivalent to the open unit disc D. We shall use the following results:

4.4.19 Result: Let G be an open connected subset of $\mathbb{C}$ **.** Then the following are equivalent.

(i) G is simply connected.

(ii) For any f in H(G) such that $f(z) \neq 0$ for all z in G, there is a function g in H(G) such that $f(z) = [g(z)]^2$.

4.4.20 Result: Let $f : D \to D$ be a one- one analytic function of D onto itself and suppose $f(a) = 0$. Then there is a complex number c with $|c| = 1$ such that

$$f = c\phi_a \text{ where } \phi_a(z) = \frac{z-a}{1-\bar{a}z} \text{ with } |a| < 1.$$

4.4.21 Open mapping theorem: Let G be a region and suppose that f is a non-constant analytic function on G. Then for any open set U in G, $f(U)$ is open.

4.4.22 Riemann mapping theorem: Let G be a simply connected region which is not the whole plane and let $a \in G$. Then there is a unique analytic function f:G→$\mathbb{C}$ having the properties:

(i) $f(a) = 0$ and $f'(a) > 0$.

(ii) f is one-one.

(iii) $f(G) = \{z : |z| < 1\}$.

Proof: First we show f is unique.

Let g be another analytic function on $\mathbb{C}$ such that g (a) = 0, $g'(a) > 0$, g is one-one and

$$g(G) = \{z : |z| < 1\} = D.$$

Then fog^{-1}: D→ D is analytic, one-one and onto. Also, $fog^{-1}(0) = f(a) = 0$.

So there is a constant c with $|c| = 1$ and $fog^{-1}(z) = cz$ for all z.

[Applying theorem (4.4.20) with a = 0]

But then, $f(z) = cg(z)$ gives that $0 < f'(a) = c\, g'(a)$.

Since $g'(a) > 0$, it follows that $c = 1$.Hence $f = g$ and so f is unique.

Now let $\Phi = \{f \in H(G)\, f \text{ is one-one}, f(a) = 0, f'(0) > 0, f(G) \subset D\}$

We first show $\Phi \neq \phi$. Since $G \neq \mathbb{C}$ so there exists $b \in \mathbb{C}$ such that $b \notin G$.

Also G is simply connected so there exists an analytic function g on G such that $[g(z)]^2 = z - b$.

Then g is one one. For this let $z_1, z_2 \in G$ such that $g(z_1) = g(z_2)$

Then, $[g(z_1)]^2 = [g(z_2)]^2$

$$z_1 - b = z_2 - b \Rightarrow z_1 = z_2$$

Thus,g is one-one.

So by open mapping theorem, there is a positive number r such that

$$B(g(a); r) \subset g(G). \qquad (1)$$

Let z be a point in G such that $g(z) \in B(-g(a);r)$

Then,

$$|g(z)+g(a)|<r$$
$$\Rightarrow \quad |-g(z)-g(a)|<r$$
$$\Rightarrow \quad -g(z) \in B(g(a);r)$$
$$\Rightarrow \quad -g(z) \in g(G) \qquad \text{[using (1)]}$$

So there exist some $w \in G$ such that

$$-g(z) = g(w)$$
$$\Rightarrow \quad [g(z)]^2 = [g(w)]^2$$
$$\Rightarrow \quad z-b = w-b$$
$$\Rightarrow \quad z = w$$

Thus, we get

$$-g(z) = g(z)$$
$$\Rightarrow \quad g(z) = 0$$

But, $z-b = [g(z)]^2 = 0$ implies $b = z \in G$, a contradiction.

Hence, $g(G) \cap B(-g(a);r) = \phi$

Let $U = B(-g(a);r)$. There is a Mobius transformation T such that

$$T(C_\infty - \overline{U}) = D$$

Let $g_1 = Tog$ then g_1 is analytic and $g_1(G) \subset D$.

Consider $g_2(z) = \dfrac{g_1(z)-\alpha}{1-\overline{\alpha}g_1(z)}$ where $\alpha = g_1(a)$.

Then g_2 is analytic, $g_2(G) \subset D$ and $g_2(a) = 0$.

Choose a complex number c, $|c| = 1$, such that

$$g_3(z) = c\, g_2(z) \text{ and } g_3'(a) > 0$$

Now $g_3 \in \Phi$ hence $\Phi \neq \phi$.

Next we assume that $\overline{\Phi} = \Phi \cup \{0\}$ (2)

Since $f(G) \subset D$, $\sup\{|f(z)|: z \in G\} \le 1$ for f in Φ. So by Montel's theorem, Φ is normal.

This gives $\overline{\Phi}$ is compact.

Consider the function $\phi: H(G) \to C$ defined as

$$\phi(f) = f'(a)$$

Hence, ϕ is a continuous function.

Since $\overline{\Phi}$ is compact, there is an f in $\overline{\Phi}$ such that $f'(a) \geq g'(a)$, for all $g \in \Phi$.

As $\Phi \neq \phi$,(2)implies that $f \in \Phi$. We show that f(G) = D. Suppose $w \in D$ such that $w \notin f(G)$.

Then the function

$$\frac{f(z) - w}{1 - \overline{w} f(z)}$$

is analytic in G and never vanishes. Since G is simply connected, there is an analytic function $h: G \to \mathbb{C}$ such that

$$\left[h(z)\right]^2 = \frac{f(z) - w}{1 - \overline{w} f(z)} \qquad (3)$$

Since the Mobius transformation $T_\xi = \dfrac{\xi - w}{1 - \overline{w}\xi}$ maps D onto D, We have $h(G) \subset D$.

Define $g: G \to \mathbb{C}$ as

$$g(z) = \frac{|h'(a)|}{h'(a)} \frac{h(z) - h(a)}{1 - \overline{h(a)} h(z)}$$

Then $g(G) \subset D,\ g(a) = 0$ and g is one- one.

Also, $$g'(\text{a}) = \frac{|h'(a)|}{h'(a)} \cdot \frac{h'(a)\left[1 - |h(a)|^2\right]}{\left[1 - |h(a)|^2\right]^2} = \frac{|h'(a)|}{1 - |h(a)|^2}$$

But, $$|h(a)|^2 = \left|\frac{f(a) - w}{1 - \overline{w} f(a)}\right| = |-w| = |\ w| \qquad \text{[as f(a) = 0]}$$

Differentiating (3), we get

$$2h(a)h'(a) = f'(a)[1 - |w|^2]$$

$$\Rightarrow \qquad h'(a) = \frac{f'(a)\left(1 - |w|^2\right)}{2h(a)} = \frac{f'(a)\left(1 - |w|^2\right)}{2\sqrt{|w|}}$$

$$g'(a) = \frac{f'(a)\left(1 - |w|^2\right)}{2\sqrt{|w|}} \cdot \frac{1}{(1 - |w|)} = \frac{f'(a)(1 + |w|)}{2\sqrt{|w|}} > f'(a)$$

This implies, $g \in \Phi$ which is a contradiction to the choice of f. Hence, we must have $f(G) = D$.

Next we prove $\overline{\Phi} = \Phi \cup \{0\}$.

Suppose $\{f_n\}$ is a sequence in Φ and $f_n \to f$ in $H(G)$.

Then, $f(a) = \lim\limits_{n\to\infty} f_n(a) = 0$

Also $f_n'(a) \to f'(a)$ so $f'(a) \geq 0$

Let z_1 be an arbitrary element of G and let $w = f(z_1)$. Let $w_n = f_n(z_1)$. Let $z_2 \in G, z_2 \neq z_1$ and K be a close disk centred at z_2 such that $z_1 \equiv K$ .Then $f_n(z) - w_n$ never vanishes on K since f is one-one But $f_n(z) - w_n$ converges uniformly to $f(z) - w$ on K as K is compact. So, Hurwitz's theorem gives that $f(z) - w$ never vanishes on K or $f(z) = w$.

If $f(z) \equiv w$ on K then f is constant function throughout G and since $f(a) = 0$, we have $f(z) \equiv 0$. Otherwise, we have f is one-one. So f' can never vanish. This gives $f'(a) > 0 \left[\because f'(a) \geq 0\right]$. So $f \in \Phi$.

Hence $\overline{\Phi} = \Phi \cup \{0\}$, which completes the proof of the theorem.